Anomalous Viscosity of Blood and the
"Summation Phenomenon"

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As an explanation for the decrease in the apparent viscosity of certain suspensions measured at high flow rates in narrow tubes, the existence of unsheared laminae was proposed by Dix and Scott Blair. They found for the flow

\[ J = \frac{\pi P}{8 \mu(\infty)} a^4 (1 + \frac{\delta}{a})^2 \]

where \( P \) is the absolute value of the pressure gradient, \( a \) the radius of the tube, \( \mu(\infty) \) the apparent viscosity of the suspension at high flow rates in tubes of large radii, and \( \delta \) the thickness of the unsheared laminae (number of unsheared laminae \( N = \frac{a}{\delta} \)). From this formula, it follows for the apparent viscosity of the suspension at high flow rates in a tube of radius \( a \)

\[ \frac{1}{\mu(a)} = \frac{1}{\mu(\infty)} \left(1 + \frac{\delta}{a}\right)^2 \]

which shows that the apparent viscosity \( \mu(a) \) decreases with the radius \( a \). Haynes recently gave a simple derivation of equation 1. We should like to point out that equation 1 has to be treated with caution. Scrutinization of the calculations shows that they are based on the assumption that the wall, together with a very thin \( \langle \delta \rangle \) layer of fluid adhering to it, can be considered as an \((N+1)\)th lamina of the suspension. This assumption is plausible and attractive, especially when we consider the extreme case of plug flow and want to use as a simple model for this complicated phenomenon the model mentioned above with \( N=1 \). Dix and Scott Blair showed how equation 1 can be reduced for this case to the equation derived by Buckingham for plug flow. Because of this assumption, however, equation 1 cannot, as sometimes is supposed, be considered to give an approximation for the flow in cases where a so-called marginal zone exists which is relatively free of particles and with a width independent of the tube radius. The existence of such a zone was demonstrated by Taylor and can in itself account for the effect of decrease in apparent viscosity with the tube radius (cf. Haynes). The existence of unsheared laminae of finite thickness in the central zone in this case tends, on the contrary, to work in the opposite direction, as will be shown. It gives an increase in the apparent viscosity, and when the radius becomes smaller, the decrease in the apparent viscosity will be smaller than it is without the presence of unsheared laminae of finite thickness.

In the presence of the marginal zone, the assumption mentioned above can no longer be used. We now have \( N \delta = a \), but \( N \delta = b < a \), and the assumption has to be replaced by the supposition that the velocity \( v(b) \) of the \( N \)th lamina is equal to the velocity \( v(b) \) of the immediately adjacent fluid of the marginal zone. When, departing from the result that is obtained in this way, one wants to ignore the existence of a marginal zone by letting \( b \) approach \( a \), the equation obtained is not our equation 1 but an equation 1' with a minus instead of a plus sign inside the brackets. So when a marginal zone exists, the presence of unsheared laminae of finite thickness actually works in the wrong direction and does not contribute to the ex-
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Figure 1
Laminations and marginal zone.

Explanation of the decrease in apparent viscosity with the radius of the tube.

Derivation of Equations
In figure 1, the laminae and the marginal zone are schematically represented. We first consider the special case that \( \delta' = \delta \). The distance of the outer boundary of the \( m^{th} \) lamina to the axis is then equal to \( m \delta \) and \( b = N \delta \). We denote the velocity in this \( m^{th} \) lamina by \( v(m \delta) \).

The contribution of the marginal zone to the total flow equals

\[
\int_{a}^{b} 2\pi r v(r) \, dr.
\]

If all laminae of the central zone possessed the velocity \( v(b) \) of the \( N^{th} \) lamina, the contribution of the central zone to the total flow would be

\[
\pi (N \delta)^2 v(b).
\]

However, the \((N-1)^{th}\) lamina has an additional velocity

\[
v((N-1) \delta) - v(N \delta).
\]

This means, that if all layers at distances \( m \delta \leq (N-1) \delta \) from the axis had the same velocity, an additional flow contribution

\[
\pi ((N-1) \delta)^2 \left[ v((N-1) \delta) - v(N \delta) \right]
\]

had to be added.

However, the \((N-2)^{th}\) lamina has an additional velocity

\[
v((N-2) \delta) - v((N-1) \delta).
\]

This means that if all layers at distances \( m \delta \leq (N-2) \delta \) had the same velocity, again an additional contribution in case

\[
\pi ((N-2) \delta)^2 \left[ v((N-2) \delta) - v((N-1) \delta) \right]
\]

had to be added; and so on.

We then obtain for the total flow

\[
J = \sum_{m=N-1}^{m=N} \pi (m \delta)^2 \left[ v(m \delta) - v((m+1) \delta) \right] + \pi (N \delta)^2 v(b) + \int_{a}^{b} 2\pi r v(r) \, dr. \tag{3}
\]

We denote the viscosity in the marginal zone by \( \mu_e \) and the coefficient of viscosity in the central region by \( \mu_e(\infty) \). \( \mu_e(\infty) \) is assumed to be equal to the apparent viscosity of this suspension in a tube with large radius. The shearing stress in the marginal zone is given by

\[
\sigma = \mu_e \frac{dv}{dr} < 0. \tag{4}
\]

In the central zone at the outer boundary of the \( m^{th} \) layer \((m<N)\),

\[
\sigma (m \delta) = \mu_e(\infty) \frac{v((m+1) \delta) - v(m \delta)}{\delta} < 0. \tag{5}
\]

We shall use equation 5 to replace the velocity difference in equation 3, but first we will derive an expression for \( \sigma \) by applying Newton's second law to a cylinder of radius \( m \delta \) and unit length. Since there is no acceleration, we obtain (\( P = \) absolute value of the pressure gradient):

\[
P = \pi (m \delta)^2 + 2 \pi m \delta \sigma (m \delta) = 0 \tag{6}
\]

In the marginal zone, this becomes \( \sigma (r) = -\frac{1}{2} Pr \). With equation 4 and the condition that \( v(a) = 0 \), equation 7 yields

\[
v(r) = \frac{P}{4\mu_v} (a^2 - r^2). \tag{8}
\]
Substituting from (5), (6), and (8) into (3) we obtain

\[ J = \frac{\pi P}{2 \mu_v(\infty)} \sum_{m=1}^{N-1} m^2 + \pi b^2 \nu(b) + \int_b^a 2 \pi r v(r) \, dr \]

\[ = \frac{\pi P}{8 \mu_v(\infty)} (N-1)^2 N^2 + \pi b^2 \frac{P}{4 \mu_v} (a^2 - b^2) + \frac{\pi P}{2 \mu_v} \int_b^a r(a^2 - r^2) \, dr \]

\[ = \frac{\pi P}{8} \left[ \frac{b^2 (b - \delta)^2}{\mu_v(\infty)} + \frac{a^4 - b^4}{\mu_v} \right]. \tag{9} \]

In the general case with arbitrary \( \delta' \), this becomes

\[ J = \frac{\pi P}{8} \left[ \frac{b^2 (b - \delta - \delta')^2 - \delta'^2 \delta^2}{\mu_v(\infty)} + \frac{a^4 - b^4}{\mu_v} \right], \tag{10} \]

where \( \mu \) denotes the apparent viscosity and is defined by

\[ \frac{1}{\mu} = \frac{1}{\mu_v(\infty)} \left[ \left( \frac{b}{a} \right)^2 \left( \frac{b - \delta}{a} \right)^2 \frac{\delta'^2}{a^2} - \frac{\delta^2}{a^2} \right] + \frac{\mu_v(\infty)}{\mu_v} \left( 1 - \frac{b}{a} \right)^4 \right]. \tag{11} \]

Here \( h = a - b \) is the width of the marginal zone, which like \( \delta \) is supposed to be independent of \( a \).

### Discussion

We assume now that the difference between \( \delta' \) and \( \delta \) is so small that the term

\[ \left( \frac{\delta'}{a} \right)^2 \left( \frac{\delta'}{a} - \frac{\delta}{a} \right)^2 \]

in equation (11) can be neglected.

1. We see from (11) that when \( \delta \) increases with \( \mu_v(\infty) \), \( \mu_v \), \( a \), and \( b \) remaining constant, \( \mu \) increases. Thus, the presence of unshaved laminae of finite thickness in the central zone tends to increase the apparent viscosity.

2. When there are no unshaved laminae \( (\delta' = \delta = 0) \), a decrease in "a" will cause a decrease in the apparent viscosity (marginal zone theory), but this decrease is smaller when unshaved laminae of finite thickness exist, owing to the presence of the term \( \frac{\delta}{a - h} \) in (11).

3. When unshaved laminae are absent \( (\delta' = \delta = 0) \), equation 10 reduces to formula 6 from Haynes paper. \footnote{Circulation Research, Volume IX, November 1961}

4. When we take \( b = a \), we obtain equation 1 except for the minus sign. We said earlier that this result is due to the difference in assumptions. In the derivation of equation 1 the Nth lamina is moving with respect to the wall. When we let (in our case) \( b \) approach \( a \), the Nth lamina is eventually in rest; equation 1 is valid for this case also if we replace \( a \) by the "effective radius" \( a - \delta \), i.e., by the distance to the axis of the outer boundary of the last—\( (N-1) \)th—lamina that is still in motion. We obtain then from (1) a result that was obtained earlier in (9):

\[ J = \frac{\pi P}{8 \mu_v(\infty)} a^4 \left( 1 - \frac{\delta}{a} \right)^2. \tag{11'} \]

5. When \( a \) (and as a consequence \( b \)) approaches infinity, \( \frac{1}{\mu} \) approaches \( \frac{1}{\mu_v(\infty)} \).

6. When we drop the supposition that the laminae are thin and want to consider the
case where there is only one lamina, the central cylinder, equation 3 reduces as follows:

\[ J = \frac{b}{2} \nu(b) + \int_{b}^{a} 2 \pi r \nu(r) dr = \frac{\pi P}{8 \mu_\nu} (a^4 - b^4). \]

In the special case \( b = a \), this reduces not to Buckingham's equation for plug flow but rather to the "equation for a plug," i.e., \( J = 0 \).

With the assumption of Dix and Scott Blair, however, the plug is not in rest but moves, separated from the wall by a thin layer of thickness \( \epsilon = a - b \ll b \) in which the velocity drops very fast from \( \nu(b) \) to \( \nu(a) = 0 \), so that

\[ -\frac{1}{2} b P = a \mu \frac{\nu(a) - \nu(b)}{\epsilon} \]

and we obtain the equation for plug flow:

\[ J = \pi b^2 \nu(b) \approx \pi a^2 \mu \frac{\epsilon}{2 \mu_\nu} a P = \pi P a^2 \mu_\nu. \]

7. Replacement of equation 3 by

\[ J = \sum_{m=1}^{N} \pi (m \delta)^2 \left\{ \nu(m \delta) - \nu(<m + 1> \delta) \right\} \]

\[ + \int_{b}^{a} 2 \pi r \nu(r) dr \]

together with the assumption that equation 5 is also valid for \( m = N \) would formally give in place of (9) the equation:

\[ J = \pi P \left\{ \frac{\delta^2 (b + \delta)^2 - \delta'^2 (\delta' - \delta)^2}{\mu_\nu(\infty)} \right\} + \left( \frac{(a^2 - b^2)^2}{\mu_\nu} \right), \]

Equation \( 9' \) reduces to equation 1 in the special case \( \delta' = \delta, b = a \). This derivation of \( (9') \) and (1), however, does not seem warranted.

**Summary**

It is shown that in the presence of a marginal zone, which is free of corpuscles and which has a width independent of the tube radius, the existence in the central zone of unshaved laminae of finite thickness, which also is independent of the tube radius, tends to increase rather than decrease the apparent viscosity when the radius of the tube becomes smaller.

**Appendix**

**Table of Symbols**

- \( a \) = Radius of the tube.
- \( b \) = Radius of the central zone.
- \( h = a-b \) = Width of the marginal zone.
- \( S \) = Thickness of the wall of the hollow cylinders of the central zone.
- \( S' = S \) = Radius of the small central cylinder within the central zone.
- \( P \) = Absolute value of the pressure gradient.
- \( \mu_0(\infty) \) = Apparent viscosity of suspension at high flow rates in a tube of large radius.
- \( X \) = Number of laminations.
- \( r \) = Distance to axis of the tube.
- \( \nu(r) \) = Velocity at the point \( r \).
- \( \sigma(r) \) = Shearing stress at the point \( r \).
- \( J \) = Flow.
- \( \approx \) = Approximately equal to.

**References**

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