A Definition for Zero Potential

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Equations are developed for the potential due to a source and sink, or dipole, inside a circle of finite resistivity surrounded by a two-dimensional infinite medium of different resistivity. When the resistivity of the external medium is allowed to increase to infinity, the resulting equations give identical results to those presented by Bayley for the dipole potential in an insulating circle. A method of defining the zero of potential for a bounded medium is proposed.

In a previous publication, image systems for a source and sink inside or outside a circle of finite resistivity surrounded by an infinite medium of different resistivity were developed. It was found that 4 systems were required, depending on whether the source and sink were inside or outside the circle, and whether the inside or outside potential was required. In this paper, the detailed derivation of the equations for a source and sink inside the circle is carried out. It is shown that a source and sink inside an insulating circle represents a limiting case in which the resistivity of the external medium is allowed to approach infinity. In the derivation, the potential is defined to be zero at infinity. The resulting equations for the completely insulating boundary differ from those given previously by an additive constant. The treatment is similar to that used by Hague in solving magnetic field problems.

In figure 1, the disk of resistivity \( \rho' \) is surrounded by an infinite two-dimensional medium of resistivity \( \rho \). The current sink, \(-I\), and source \(+I\), are located at distances \( a \) and \( b \) from the center of the circle. In similar problems in electrostatics, images are located outside the circle at the inverse points. We assume, therefore, that the current inside the disk is due to \(+I\) and \(-I\), and to images of strength \(+J\) and \(-J\) at \( R^2/b \) and \( R^2/a \), and that the entire region has the resistivity \( \rho' \). It is also assumed that the current outside is due to a source and sink at \( b \) and \( a \) of strengths \( +N \) and \(-N\), and that all space has the resistivity \( \rho \). The potential \( V_i \) at any point inside the disk is then

\[
V_i = \frac{\rho' I}{2\pi d} \ln \frac{r_1}{r_2} + \frac{\rho' J}{2\pi d} \ln \frac{r_2}{r_4} + C_1
\]

and the potential at any point outside is

\[
V_o = \frac{\rho N}{2\pi d} \ln \frac{r_3}{r_4} + C_2
\]

In the above, \( d \) is the thickness of the conductive material, \( C_1 \) and \( C_2 \) are arbitrary constants, and \( r_1 \) to \( r_4 \) are the distances from the sink, source, image of sink, and image of source to the point at which the potential is required, hereafter designated the "search point." We now stipulate that the potential shall be equal to zero at infinity. In equation (2), as the search point approaches infinity, \( r_1/r_2 \rightarrow 1 \). Hence, at infinity, \( V_o = C_2 \), and therefore \( C_2 = 0 \).

At the boundary between the two media, we must have

\[
V_o = V_i \text{ (at boundary)}. \tag{3}
\]

For the point \( P \) in figure 1, therefore

\[
\rho' I \ln \frac{r_1}{r_2} + \rho' J \ln \frac{r_2}{r_4} + 2\pi d C_1 = \rho N \ln \frac{r_3}{r_4} \tag{4}
\]

It can be shown by similar triangles that, at the boundary

\[
\frac{r_2}{r_1} = \frac{b}{a} \tag{5}
\]

Hence, from equations (5) and (4)

\[
\rho' (I + J) = \rho N \tag{6}
\]

\[
C_i = \frac{\rho' J}{2\pi d} \ln \frac{a}{b} \tag{7}
\]
Next, it is necessary to equate normal currents at the boundary, since $i_n$ is continuous. The current intensity due to a source $+I$ at a point $r$ cm. away is

$$i = \frac{I}{2\pi r^2}$$

(8)

The current intensity vector, $i$, is defined as the current flowing through a unit surface area of the electrolyte. Thus $2\pi dr$ is the surface area of a circle of electrolyte of radius $r$ and thickness $d$. In figure 1, the currents must be multiplied by the cosines of the angles shown to find the components normal to the boundary. The directions of the current intensity vectors are shown by the arrows. Outward flowing currents are defined as positive. Hence

$$L \cos \theta_2 - \frac{J \cos \theta_1}{r_1} + J \cos \theta_2 = \frac{N \cos \theta_2 - N \cos \theta_1}{r_2}$$

(9)

By the use of similar triangles, it is possible to show that

$$\frac{\cos \theta_1}{r_1} = \frac{\cos \theta_2}{r_2} = \frac{1}{R}$$

(10)

Substituting (10) and (11) in (9), we get

$$I - J = N$$

(12)

Solving (10) and (12)

$$J = \frac{\rho - \rho'}{\rho + \rho'} \cdot I$$

(13)

$$N = \frac{2\rho'}{\rho + \rho'} \cdot I$$

(14)

Finally, combining (1), (7), and (13)

$$Y_i = \frac{\rho' I}{2\pi d} \left[ \ln \frac{r_1 + \rho - \rho'}{r_1} + \ln \frac{r_2 + \rho + \rho'}{r_2} \right]$$

(15)

Defining

$$G = \frac{\rho'}{\rho}$$

(16)

$$Y_i = \frac{\rho' I}{2\pi d} \left[ \ln \frac{r_1 + 1 - G \frac{a r_2}{b r_2}}{r_1} + \ln \frac{1 + G \frac{a r_2}{b r_2}}{r_2} \right]$$

(17)

This is equivalent to equation 1 of reference 1. The equation holds for all values of $G$ from zero to infinity. Thus for the case of a circular boundary

$$r = \frac{a}{\sqrt{1 - G \frac{a r_2}{b r_2}}}$$

(18)

Here $a$ and $b$ are constants. The solution is valid for the case of a circular boundary.
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lamina of electrolyte in a metal ring, \( G \to \infty \), and (17) becomes

\[
V_i = \frac{\rho I}{2\pi a} \ln \left[ \frac{r_1}{r_2} \right] - \frac{\rho I}{2\pi a} \ln \left[ \frac{r_1 r_2 b}{r_2 a} \right]\]

(18)

The images are located as in figure 1, but have the opposite sign. Thus the image of the source is a sink, and the image of the sink is a source. If, on the other hand, the boundary is completely insulating, \( G = 0 \), and (17) becomes

\[
V_i = \frac{\rho I}{2\pi a} \ln \left[ \frac{r_1 r_2 a}{r_2 r_1 b} \right]
\]

(19)

The images now have the same signs as the real source and sink. Equation (19) can be written

\[
V_i = \frac{\rho I}{2\pi d} \left[ \ln \left( \frac{r_1 r_2 a}{r_2 r_1 b} \right) + \ln \left( \frac{a}{b} \right) \right]
\]

(20)

This is identical with equation (3) of reference 2, except for the additive term, \( \ln (a/b) \), which is constant for a given source and sink.

**Dipole Form of Equation.** Once the solution for the potential distribution due to a source and sink has been found, the dipole form of the solution can be derived by means of an expansion. Figure 2 shows the method for \( r_1/r_2 \). By dropping perpendiculars from \( r_1 \) and \( r_2 \) to \( c \), it is evident that, approximately

\[
r_1 = r + \frac{D}{2} \cos \theta_i
\]

\[
r_2 = r - \frac{D}{2} \cos \theta_i
\]

(21)

\[
\ln \left( \frac{r_1}{r_2} \right) = \ln \left( \frac{c + \frac{D}{2} \cos \theta_i}{c - \frac{D}{2} \cos \theta_i} \right) = \ln \left( \frac{1 + \frac{D \cos \theta_i}{c}}{1 - \frac{D \cos \theta_i}{c}} \right)
\]

(22)

The last term of equation (22) is a standard expansion (for example, Peirce and Foster, no. 843) and gives

\[
\ln \left( \frac{r_1}{r_2} \right) = \frac{D \cos \theta_i}{c} + \frac{1}{12} \left( \frac{D \cos \theta_i}{c} \right)^3 + \frac{1}{80} \left( \frac{D \cos \theta_i}{c} \right)^5 + \cdots
\]

(23)

For values of \( D \) small with respect to \( c \), all terms but the first can be neglected. Considering that \( \ln (a/b) \) is equal to the potential at the center of the circle, an exactly similar procedure gives

\[
\ln \left( \frac{a}{b} \right) = \frac{D \cos \omega}{h}
\]

(24)

where \( h \) is the distance of the dipole center from the center of the circle, and \( \omega \) is the angle between the dipole axis and \( h \). From similar triangles it can be shown that the pole separation \( D' \) between the images of the source and sink is given by

\[
D' = \frac{R^2}{ab} D \approx \frac{R^2}{h^2} D
\]

(25)

The expansion then gives

\[
\ln \left( \frac{r_1}{r_2} \right) = \frac{R^2}{h^2} \frac{D \cos \theta_i}{c} + \cdots
\]

(26)

The dipole form of equation (19) is then

\[
V_i = \frac{\rho M}{2\pi d} \left( \frac{\cos \theta_i}{c} + \frac{R^2 \cos \theta_i}{h^2 c'} + \frac{\cos \omega}{h} \right)
\]

(27)

where \( M \) is the dipole moment, defined by

\[
M = ID
\]

(28)

The last term in equation (27) is constant for a specific dipole orientation. Apart from this...
constant, the equation is identical to equation (7) of reference 2. It should be pointed out that equations (19) and (27) hold for any point inside or on the boundary of the circle. The cosine form of equation (17) for any resistivity values is

\[ V_i = \frac{\sigma M}{2\pi d} \left[ \frac{\cos \theta_i}{c} + \frac{1 - G R^2 \cos \theta_1}{1 + G h^2 c'} \right] \]

(29)

A summary of the potential values for the general as well as special cases is shown in figure 3.

In these equations, the algebraic signs of the images are ignored. Thus \( r_3 \) and \( r_4 \) are defined as the distances from \( P \) to the image of the sink and the image of the source, regardless of whether the boundary is insulating or conducting. Similarly, the angle \( \theta_2 \) is defined on the assumption that the image dipole vector is antiparallel with the real dipole vector.

<table>
<thead>
<tr>
<th>Type of boundary</th>
<th>Finite resistance</th>
<th>Insulating</th>
<th>Metal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of Resistivity ratio ( G = \rho_p )</td>
<td>Any</td>
<td>1.0</td>
<td>Very large</td>
</tr>
<tr>
<td>Potential at any interior point.</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
</tr>
<tr>
<td>Source-sink form</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
</tr>
<tr>
<td>Potential on boundary of disk.</td>
<td>( 2A' \ln \frac{d}{b} )</td>
<td>( 2A' \ln \frac{d}{b} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>Source-sink form</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
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<tr>
<td>Potential at center of disk.</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
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<tr>
<td>Source-sink form</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
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<tr>
<td>Potential at any interior point.</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
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<tr>
<td>Dipole form</td>
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<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
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<tr>
<td>Potential on boundary of disk.</td>
<td>( 2A' \ln \frac{d}{b} )</td>
<td>( 2A' \ln \frac{d}{b} )</td>
<td>( 0 )</td>
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<td>Dipole form</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
</tr>
<tr>
<td>Potential at center of disk.</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
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<td>Dipole form</td>
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<td>( A' \ln \frac{d}{b} )</td>
<td>( A' \ln \frac{d}{b} )</td>
</tr>
</tbody>
</table>

1. At points corresponding to boundary of any circle.
2. At a point corresponding to the center of any circle.

\[ \varepsilon' = \frac{\rho' \cdot d}{2\pi d} \quad \rho = \frac{p' \cdot d}{p} \]

**Fig. 3.** Potential in a circular disk due to a source and sink or dipole inside the disk.
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small motor-generator of 50 volt-amperes capacity. The transformer secondary was connected to ground. The transformer dipole was energized by 400 cycles AC taken from a Saline solution was used as the electrolyte. The artificial clipole consisting of two platinum wires in a circular plastic dish with a radius of 7.5 cm. The generator output was applied to a transformer with a balanced secondary; the center-tap of the output voltage was about 300 volts each side of the insulating boundary.

scope, which had an input impedance of 4 megohms. A separate constant low voltage from electrodes was applied directly to the balanced input of a sensitive dual-beam cathode ray oscilloscope. The reference electrode was placed near, but not on, the dipole traverse axis. Its potential was arbitrarily defined as being positive or negative depending on whether it was in phase or out of phase with this voltage. The search and reference electrodes were platinized-platinum. For experiments involving a highly conducting boundary, a silver-plated ring was inserted in the disk. Potentials were measured with respect to this ring.

The results are shown in table 1. Column 6 shows the averages of the measured values, and column 5 the averages of the values calculated by the three equations. No attempt was

<table>
<thead>
<tr>
<th>Point</th>
<th>Source</th>
<th>Dipole</th>
<th>Bayley</th>
<th>Ave. calc.</th>
<th>Ave. $V_{mn}$</th>
<th>$V_{mn}$</th>
<th>$V_{mn}$</th>
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</table>

With these conventions, signs are determined by the factor $(1 - G)/(1 + G)$.

METHODS

The equations were checked by means of an artificial dipole consisting of two platinum wires in a circular plastic dish with a radius of 7.5 cm. Saline solution was used as the electrolyte. The dipole was energized by 400 cycles AC taken from a small motor-generator of 50 volt-amperes capacity. The generator output was applied to a transformer with a balanced secondary; the center-tap of the secondary was connected to ground. The transformer output voltage was about 300 volts each side of center, but this was dropped to less than 1 volt at the electrodes by means of high series resistance. Constant-current feed was used, and a rectifier type AC milliammeter was used to monitor the electrode current.

The voltage between the search and reference electrodes was applied directly to the balanced input of a sensitive dual-beam cathode ray oscilloscope, which had an input impedance of 4 megohms. A separate constant low voltage from the transformer was applied to the other oscilloscope beam. Polarity of the measured potential was
made to achieve high accuracy in this series of experiments, since the objective was mainly to check the validity of the equations developed. The milliammeter was an ordinary AC rectifier type panel meter, and the peak-to-peak voltage deflections were read visually from the oscilloscope screen. No corrections were made for the finite size of the dipole wires. In making the calculations, distances and angles were measured with a centimeter scale and a protractor. Fluid depth was measured accurately with a spherometer.

The theory shows that if a metal boundary is used, its potential is zero, regardless of the internal position of the dipole. Accordingly, in experiments in which this type boundary was used, potentials were measured with respect to this ring. For any internal dipole position, potentials appeared to be zero along the transverse axis. Again, the equations developed were found to be correct.

Space does not permit inclusion of all the data taken, but one case is of special interest. For a centric dipole, equation (19) can still be used, with $a/b = 1$. Equation (27) can not be used in the form shown, however, since as the dipole approaches the center, the third term approaches infinity. In order to find out how close to the center equation (27) is still applicable, measurements were made of the potential at a fixed point as a radial dipole on the x-axis was moved closer to the center (fig. 5).

For locations of the dipole center from $X = 5.5$ to 0.25 cm., potentials were calculated by means of equation (27). For the dipole exactly at the center, the potential was calculated by the following equation, given by Wilson, MacLeod, and Barker.

$$I = M \cos \theta \left( \frac{1}{r} + \frac{R}{R^2} \right)$$

where $\theta$ is the angle which the dipole makes with the $+X$ axis, $r$ is the distance of measured point from center of circle, and $R$ is the radius of circle.

It is evident from figure 5 that equation (30) represents a limiting case of equation (27) as the dipole approaches the center. This has also been confirmed mathematically by Dr. Bayley, in a private communication.

DISCUSSION

In experiments with an artificial dipole in a two-dimensional lamina, Bayley found a discrepancy between his results and equations previously published by the author. Measurements have shown that values obtained using equations (3) and (7) of that publication differ from values found from Bayley’s equation by a constant, for a given dipole orientation. The equations given previously are identical to equations (19) and (27) except for the additive constants, $\ln a/b$ and $(\cos \omega)/\kappa$. In any field problem, the solution is given by a term which represents the variation of potential with position, called the “potential function,” plus a constant term. This arises from the fact that potential is defined as an integration of field intensity, and every integration involves a “constant of integration.” The value of the constant is arbitrary and may be given any convenient value. The fact that the constant is arbitrary may be viewed from another standpoint. The field strength, $E$, which has a definite value at any
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point, may be considered to be the gradient, or derivative, of the potential at that point. Since the derivative of a constant is zero, vector \( \mathbf{K} \) will have the same value regardless of the value of the constant term. The equations in reference 2 give the value of the potential function, with the constant term assigned the value zero. It is necessary to calculate the potential of the reference electrode also and to subtract this value from the search probe potential in order to agree with the measured values.

In order to test the correspondence between equations (19), (27) and Bayley's equation, the potential at a point due to an eccentric dipole in a circle was calculated by all three equations. This was done very carefully, and all distances and angles were calculated from the coordinates. It was found that all three equations gave exactly the same result.

Zero of Potential. Burger has discussed the problem of the zero of potential, and has pointed out that there is no such thing as an absolute zero of potential. On the other hand, we are entitled to arbitrarily define the zero of potential, and provided that there is a general agreement on the definition, we can thenceforth speak of the zero of potential without ambiguity. For an unbounded medium, most authors specify that the potential shall be set equal to zero at infinity. In the double-layer electrolytic tank, the electrode at the center of the bottom layer can be considered to be located at infinity, and zero potential points in the top layer therefore agree with this definition. For a volume conductor bounded by an insulating surface the problem is much more difficult, since the field does not extend to infinity. In the derivation of equation (19), it was assumed first that the circle containing the source and sink was surrounded by an infinite two-dimensional layer of conducting fluid. The potential outside was set equal to zero at infinity. The resistivity of the external medium was then allowed to increase to a very high value, and the limiting expression for the internal potential was found. The zero potential points in the internal medium should then conform with the definition of zero potential at infinity. This procedure therefore provides a method for defining zero potential in a bounded medium.

In accordance with this system all equations derived above and listed in figure 3 give the potential with respect to zero. Bayley's equation also agrees with this definition, since measurements and calculations show that this gives the same results as (19) and (27). Zero potential for other figures, such as the ellipse, sphere, and cylinder could be determined in a similar way. The method could also be applied experimentally to the human or animal body. Immersion experiments could be carried out in a large volume of liquid with a remote reference electrode. The zero potential points on the body could then be measured, using bathing liquids of successively higher values of resistivity. Graphs of zero point location as a function of fluid resistivity could then be extrapolated to the limiting case of completely nonconducting fluid. One could of course make immersion experiments directly with a fluid of very high resistivity provided that experimental errors do not become excessive. A few such experiments might be sufficient to check less elaborate theoretical concepts of zero potential. An advantage of this system would be that one would have the same zero of potential for a region surrounded by a boundary with finite resistance as for a completely insulating boundary.

Potential at Center of Circle. Equation (17) may be solved for the potential at the center of the circle by putting

\[ r_1 = a, \quad r_2 = b, \quad \rho = \frac{R^2}{a^2}, \quad \rho = \frac{R^2}{b^2} \]

Then

\[ V = \frac{\rho l}{2\pi d} \ln \frac{a}{b} \]

The corresponding dipole equation is

\[ V = A \cdot D \cdot \frac{\cos \omega}{h} \]

It is seen that these equations hold for any type of circular boundary, or for no boundary at all. The potential at the center is independent of the boundary, provided that the current is kept constant. It is interesting that the addi-
tive constant in equation (19) or (27) is equal to the potential at the center of the circle. This is not the case for other values of \( G \), however.

**Potential on Boundary.** Substituting (5) into (17), on the boundary

\[
V = \frac{2}{1 + G} A' \ln \frac{r_1}{r_2}
\]  

(33)

As the boundary is varied from highly conducting metal, \( G \gg 1 \), to a complete insulator, \( G = 0 \), \( V \) varies from zero to \( 2 A' \ln r_1/r_2 \). In the latter case, the potential is twice the value it would be at the same point if no boundary were present. This has also been shown to be true for an infinite straight-line boundary.

**Potential on Transverse Axis.** For any point on the transverse axis of the dipole, \( r_1 = r_2 \). Equation (33) shows that the potential is zero at the intersection of the transverse axis with the boundary, for any value of \( G \). For interior points along the transverse axis, the potential depends on the values of \( r_3, r_4, a, \) and \( b \), as well as the value of \( G \). Our measurements show that in practice the potentials along the transverse axis are not far from zero. For the unbounded case, the potential is zero at all points along the transverse axis.

The best location for the reference probe is therefore at the boundary on the transverse axis, when making measurements with an artificial dipole. The reference probe will be at zero potential only when it is on the transverse axis of the dipole. If the dipole is moved or rotated, the potential of the reference probe must be calculated and subtracted from the calculated potential of the search probe in order to obtain an agreement with measured values. In experiments with a moving dipole, a reference which is at zero potential (as defined above) for any dipole orientation is supplied by the bottom-layer electrode in the double-layer tank, or by the averaging network of Bayley.

**Summary**

Equations are developed for the potential due to a source and sink or dipole in a circular region having a finite value of electrical resistivity surrounded by an infinite region of different resistivity. The equations are completely general and hold for all values of resistivity. An external region of very high resistivity is equivalent to a circular insulating boundary; a boundary of essentially zero resistivity can be simulated by a metal ring. The equations for the insulating boundary give the same results as Bayley's equation.

A method for defining the zero of potential for bounded regions is proposed. The method consists of the following steps: 1. Obtain the solution for the potential of the dipole in the region, assuming that the region is surrounded by an infinite medium of finite resistivity. 2. Adjust the arbitrary constants so that the potential is zero at infinity. 3. Allow the resistivity of the external medium to increase to a very large value, and obtain the solution for the limit. 4. Points of zero potential in this solution are then defined as required.

**Appendix**

**Some Mathematical Magic—Transformation of a Circle into an Infinite Straight Line**

Two-dimensional problems involving boundaries can sometimes be simplified by a conformal transformation of the region, so that the boundary assumes a simpler shape. This procedure, although frequently used in mathematics in complex variable theory, has only rarely been applied to problems in biology and medicine. Sugi used a transformation in studying the injury potential of muscle. In figure 6A, the point \( P \) has the coordinates \( x, y \). The complex variable \( z \) is defined by the equation

\[ z = x + jy \]

\( j \) indicates a rotation of 90° from the \( x \)-axis. Each point in the plane will then have a definite value of \( z \) corresponding to each \( x \) and \( y \). If \( w = u + jv \) is another complex variable, and if \( w \) is some function of \( z \), then for each value of \( z \) there will exist a corresponding value of \( w \). Thus if

\[ w = 2z \]

or

\[ u + jv = 2(x + jy) \]

then

\[ u = 2x \]

\[ v = 2y \]

This is shown in figure 6B.
A DEFINITION FOR ZERO POTENTIAL

FIG. 7. Conformal transformation of circle into an infinite straight line. Corresponding lines in the two diagrams have the same marking. The circle becomes the $u$-axis in the $w$-plane. The area inside the circle becomes the infinite region above the $u$-axis. The dipole has a mirror image in this axis.
and \( w \) can also be thought of as vectors from their respective origins to the point. If there is a circle in the \( z \) plane of radius \( R \) and center at \( z = 0 \), the transformation
\[
w = j \frac{R - z}{R + z}
\]
transforms the circle into an infinite straight line in the \( w \) plane lying along the \( u \)-axis. In figure 7, corresponding points have the same letter. The semicircle \( A, D, G \) is transformed into the \( +u \)-axis, and, the semicircle \( A, K, G \) goes into the negative \( u \) axis. The point \( G \) is considered to be at both \( \pm \) infinity. The \( Y \)-axis \( D, P, S, AA, K \) becomes a semicircle in the \( w \) plane. The point \( A \) becomes the origin of coordinates, and the portion of the \( x \)-axis inside the circle becomes the positive \( v \) axis. The point \( G \) is again at infinity. All of the area inside the circle becomes the infinite region above the \( u \)-axis. All of the area outside the circle becomes the infinite region below the \( u \)-axis. The source and sink inside the circle take up positions above the \( u \)-axis. Although the strengths of the transformed source and sink remain the same, their relative position and the distance between them is different in the \( w \)-plane. Hence a dipole in the \( z \) plane will, in general, have a different direction and magnitude in the \( w \)-plane. The dipole transverse axis, \( CC \) to \( KK \), becomes a curved arc in the \( w \)-plane, intersecting the boundary at two different points.

In the derivation above it was assumed that the image of the source in the circle lay at the inverse point. Although the fact that this assumption led to the correct equations can be considered a type of proof, a direct proof can be obtained by means of the transformation. In figure 7 Bottom, the image of the source \( +I \) in the \( u \)-axis (assuming this to be insulating) is an equal source \( +I' \) at an equal distance from the axis. By locating the point corresponding to \( +I' \) in the \( z \) plane, it is found that this lies along the radius through \( +I \) at a distance of \( R^2/y \) from the center. The same procedure can be carried out for the sink.

Similarly, the image of the dipole in the \( u \)-axis is an antiparallel dipole as shown. The image dipole is then determined in the \( z \) plane by calculating the location of its center, and the magnification and rotation of the dipole in the inverse transformation,
\[
z = R \frac{j - w}{j + w}
\]
It is found that, in the \( z \)-plane, the image dipole is located along the radius through the real dipole-center, at a distance of \( R^2/c^2 \) from the center of the circle. Its magnitude is \( R^2/c^2 \) times the magnitude of the real dipole, and it is antiparallel with the real dipole, relative to the common radius. A summary of the relationships which exist in the transformation are given below, without proof. Polar form
\[
\begin{align*}
    z &= r \theta = x (\cos \theta + j \sin \theta), \\
w &= m \phi = m (\cos \phi + j \sin \phi)
\end{align*}
\]
For the circular boundary,
\[
\begin{align*}
    z &= \tan \frac{\theta}{2} = y = \frac{2Rv}{1 + u^2} \\
w &= v = 0 \\
\end{align*}
\]
For any point
\[
\begin{align*}
    u &= \frac{y}{(R + x)^2 + y^2} \\
v &= \frac{R^2 - r^2}{(R + x)^2 + y^2} \\
x &= \frac{R(1 - u^2 - v^2)}{u^2 + (v + 1)^2} \\
y &= \frac{2Ru}{u^2 + (v + 1)^2}
\end{align*}
\]
Magnification of dipole
\[
\frac{2R}{(R + x)^2 + y^2}
\]
Rotation of dipole
\[
\tan^{-1} \left[ \frac{(R + x)^2 - y^2}{2y(R + x)} \right]
\]
The magnification of the dipole is the number by which the dipole strength, or vector length, in the \( z \) plane must be multiplied in order to obtain its strength in the \( w \) plane. The rotation of the dipole is the angle through which it must be rotated from its position in the \( z \) plane in order to determine its correct direction in the \( w \) plane. Both of these factors are functions of position.

The transformation process may be visualized as follows: Rotate the circle \( 90^\circ \), so that the point \( A \) is at the bottom, and point \( G \) at the top. Move the circle up a distance equal to its radius. Cut the circle on each side of the vertical line at \( G \). Now imagine that all lines are made of rubber and can be stretched. Grasp the right semicircle near \( G \) and pull this out to \( + \) infinity to the right. Similarly pull the left semicircle to the left to minus infinity; also pull the vertical line upwards to infinity. The lines \( KSD \) and \( CC, GG, KK \) must be stretched into curved lines, as shown.

**SUMMARY IN INTERLINGUA**

Es disveloppate equationes pro le potential de un fonte e dipolo in un region circular a valor finite de resistivitate electric que es circumdate de un region infinite de resistivitate differente. Le equationes es completamente
A definition for zero potential

A definition for zero potential. A region external of high resistivity is equivalent to an insulating border. A border of resistivity essentially zero can be simulated by a metal ring. The equations for the insulating border produce the same results as the equation of Bayley.

A method is proposed to define the zero potential for bordered regions. It consists of the following steps: (1) Obtain the solution for the potential of a dipole in the region, with the assumption that the region is surrounded by an infinite medium of finite resistivity. (2) Adjust the arbitrary constants so that the potential is zero at infinity. (3) Permit the resistivity of the external medium to increase to a high value and obtain the solution for the limit. (4) Points of potential zero in this solution are then defined according to what is required.

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A Definition for Zero Potential
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