Constitutive Relations and Finite Deformations of Passive Cardiac Tissue II: Stress Analysis in the Left Ventricle

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We present a new approach for estimation of transmural distributions of stress and strain in the equatorial region of a passive left ventricle. We employ a thick-walled cylindrical geometry, assume that myocardium is incompressible, and use a three-dimensional constitutive relation that yields a material symmetry consistent with observed transmural variations in muscle fiber orientations. Moreover, we consider finite deformations including inflation, extension, twist, and transmural shearing and suggest a new method for determination of the requisite deformation parameters directly from experimental strain data. We show representative transmural distributions of stress and strain, and perform a parametric study to illustrate differing predictions of stress induced by varying boundary conditions, muscle fiber orientations, or modes of deformation. Our analysis can be used to guide and check future predictions of cardiac stresses, and to guide experimentalists by suggesting the accuracy of measurements essential for stress analysis in the heart. (Circulation Research 1989;65:805-817)

Quantification of the mechanical stresses in the heart remains a subject of intense interest. Mechanical stresses can, for example, affect coronary blood flow, myocardial oxygen consumption, overall cardiac function, and the rate and extent of hypertrophy in certain pathological conditions. These stresses cannot be measured directly in the intact heart, but must be inferred from biomechanical analyses. Thus, most reported inferences of stresses in the heart have been obtained from either finite-element-based calculations or from closed-form analytical solutions.3-6

Finite-element methods are numerical approximation techniques that can, in theory, account for the geometric irregularities, complex boundary conditions, large deformations, solid-fluid interactions, material heterogeneities, and nonlinear anisotropic material behavior inherent to the heart. This approach may, in fact, be the only practical tool for accurate analysis of regional stresses in the intact heart. At present, however, realistic predictions of the three-dimensional state of stress in the heart are limited by a lack of sufficient experimental data, particularly information on the material properties of myocardium. Consequently, results obtained using finite-element methods must be interpreted with caution.

Because of the lack of complete data and the sometimes prohibitive cost and effort in performance of finite-element calculations, closed-form analytical solutions can continue to provide valuable insight into cardiac mechanics, as long as one views the predictions within the framework of the underlying assumptions. Such solutions presented before 1980 generally considered the heart to be homogeneous, however, and did not account for the known distributions of muscle fibers throughout the walls.3-5 Thus, the utility of the associated predictions was questionable. Recent investigators, using a thick-walled cylindrical geometry to model the left ventricle, have attempted to take fiber angles into account but have assumed that myocardium consists solely of "slippery" muscle fibers embedded in an inviscid fluid.7-14 Hence, it was tacitly assumed that myocardium can bear loads only in the direction of the muscle fibers. This assumption, however, is inconsistent with recent experimental data on noncontracting tissue15; therefore, the usefulness of predictions based on these cylindrical models of passive left ventricular mechanics is also questionable.

In this paper, we present a new analytical approach for calculation of transmural distributions...
of stress and strain in the equatorial region of a noncontracting left ventricle. Like other investigators, we base our analysis on a thick-walled cylindrical geometry, a reasonable representation of the distribution of muscle fibers in the left ventricle,12,16 incompressibility of the tissue, and large deformations including inflation, extension, and twist. In contrast with previous studies, however, we use a recent constitutive description of myocardial properties derived from biaxial data17 and account for transmural shearing strains, which were recently identified in canine hearts.18 Our new constitutive relation treats myocardium as transversely isotropic with respect to the muscle fiber directions and accounts for both the "fiber" and the previously neglected "cross-fiber" stiffness13 of the tissue. Finally, whereas the parameters that define the deformation are often inferred from assumed loading conditions,8,11-13 we calculate these parameters directly from available experimental data18 and by satisfaction of the equations of equilibrium.

We show transmural distributions of stress and strain in the equatorial region of a representative passive left ventricle. We also examine changes in the predicted stresses induced by different distributions of muscle fibers, varying amounts and types of deformations, and differing boundary conditions. Because of the current lack of sufficient experimental data, however, our predictions of stress distributions, like all others, should be viewed cautiously with regard to physiological implications. Nonetheless, our analysis is reasonably general, and can be refined to include new information as it becomes available. Finally, our analysis can be used to guide and check future calculations of cardiac stresses, and to guide experimentalists by suggesting the requisite accuracy of measurements that are needed for more complete and realistic assessments of stress in the heart.

Methods

Preliminary Considerations

We will first address several issues that are fundamental to our analysis, including the applicability of the continuum hypothesis and the choice of constitutive relations, geometry, deformations, and boundary conditions.

Continuum approach. Some investigators, using measurements of gross forces and deformations, have attempted to calculate stresses and extensions in individual muscle fibers or sarcomeres.7,8,11-13,19 To accurately determine these stresses and extensions, however, one must know the precise details of the microstructure, including the number, orientation, distribution, diameter, and properties of the tissue constituents and their interactions. Since this information is not available, there is no way to discriminate between the loads borne by individual muscle fibers and those borne by neighboring tissue components; hence, estimates of stress in individual muscle or collagen fibers are likely to be misleading.

We, like many others, employ an approach that is useful when one considers gross mechanical behavior of a material, namely, the relationship between locally averaged stresses and strains at any point within a body. This continuum approach is reasonable when the size of the constituents that make up the material and the distances between them are orders of magnitude smaller than the dimensions of interest. In the canine heart, for example, the left ventricle is roughly 1-2 cm thick, whereas the diameters of muscle fibers and intramuscular collagen and the interfiber distances are on the order of 0.1-3 μm.20 Hence, for the usual considerations of, for example, transmural distributions of stress, the continuum hypothesis seems reasonable.

Constitutive relations. Equations that characterize a material and its response to applied loads are called constitutive relations since they describe gross behavior due to the internal constitution of the material. Recent biaxial experimental data reveal that potassium-arrested myocardium can support loads both transverse to and in the direction of the muscle fibers, and that the tissue behavior is nonlinear and anisotropic.15,21 Thus, existing characterizations of myocardium as a fiber-reinforced, fluid-filled structure,8,10-13 or as an isotropic material,22-26 are inappropriate descriptors of the tissue behavior.

Consequently, we employ a constitutive relation derived directly from biaxial data and conveniently expressed in terms of a pseudostrain-energy function, \( W(I_1, \alpha) \), as17

\[
W = c(e^{a(I_1-3)} - 1) + a(e^{b(\alpha - 1)^2} - 1)
\]  

where \( c, b, A, \) and \( a \) are pseudoelastic material parameters derived from multiaxial data, and \( I_1 \) and \( \alpha \) a stretch ratio in the direction of a muscle fiber, are coordinate invariant measures of the deformation. Stress-strain relations are obtained from Equation 1 by taking derivatives of \( W(I_1, \alpha) \) with respect to the appropriate measures of the deformation (see "Appendix").

Geometry. A thick-walled sphere does not closely approximate the actual geometry of the heart or left ventricle. Moreover, the deformation of a sphere into another sphere induced by uniform pressures on its surfaces implicitly requires either isotropic or transversely isotropic (with respect to radial direction) material symmetry. Thus, spherical models are not very useful for estimation of stresses and strains in the heart even though the solutions are straightforward.22-24,27 Conversely, a prolate spheroid resembles the geometry of the left ventricle, but introduces substantial mathematical complexity, particularly when large deformations are considered. Thus, simplifying assumptions are often introduced, such as treatment of the ventricular wall as a series of uncoupled thin layers described by Laplace's relation,28,29 a conjecture that remains questionable.18
Alternatively, a thick-walled cylindrical geometry is mathematically tractable even when one considers certain complex finite deformations of an incompressible, anisotropic, nonlinear material. Thus, like many investigators, we employ a cylindrical geometry to study the passive left ventricle. We emphasize, however, that a cylindrical geometry can, at best, approximate the left ventricle only in a small annular region (Figures 1A and 1B; see also Figure 2 in Streeter\textsuperscript{16}) and not over its entirety.\textsuperscript{28} Although left ventricular cross sections are admittedly not circular near the equator, this approximation seems warranted if one is not interested in detailing the stresses and strains in the trabeculae and other interstices near the endocardial surface. Finally, consideration of the equatorial region as cylindrical is consistent with finite-element predictions that the variations in stress are primarily radial in the equatorial plane, but are both radial and axial nearer the apex and base.\textsuperscript{30,31}

Muscle fiber orientations. The data of Streeter\textsuperscript{16} reveal that the orientations of the muscle fibers vary nonlinearly from the endocardium to the epicardium in the equatorial region of the canine left ventricle. Moreover, the muscle fibers appear to follow regular helices on constant radial surfaces. Thus, Tozeren\textsuperscript{12} suggested the following equation to represent these reported distributions of muscle fiber orientations, $\Phi$, as

$$\Phi(R) = \Phi_e \left[ \frac{2R - (R_e + R_i)}{|R_e - R_i|} \right]^3$$

(2)

where $\Phi_e$ is the value of $\Phi$ at the epicardium (e.g., $-60^\circ$ to $-90^\circ$) and $R_i$ and $R_e$ denote unloaded endocardial and epicardial radii, respectively. Although this equation only approximates the actual muscle fiber orientations in the heart, we employ it to obtain representative predictions of transmural stress. Finally, a linear distribution of $\Phi(R)$ results from Equation 2 if the exponent on the term in $\{ \ldots \}$ is changed from 3 to 1.\textsuperscript{11,12}

Deformations. Attempts to rigorously quantify the complex deformations experienced by the heart have been reported only recently.\textsuperscript{18,32–34} These data reveal that large in-plane (i.e., in constant radial surfaces) and transmural shearing strains accompany finite extensional strains throughout the cardiac cycle.\textsuperscript{18} Transmural shearing strains correspond to the $t_{rr}$ and $t_{rt}$ stresses, and the in-plane shear strain corresponds to $t_{ee}$ (Figure 1B). Whereas the presence of transmural shearing strains has been neglected in previous cylindrical models of the left ventricle, we consider a deformation that is consistent with the existence of all six reported components of Green strain.\textsuperscript{18}
Stress boundary conditions. Uniform pressures are normally assumed to act on the inner and outer surfaces of the heart (Figure 1C). The pressure at the inner surface, \( P_i \), is the intraventricular cavity pressure, whereas the external pressure, \( P_o \), is usually assumed to be zero.\(^8,11-13\) Neglect of \( P_o \) is suspect, however, in light of increasing data and speculation on the mechanical role of the parietal and visceral pericardia.\(^35-37\) Moreover, because the left ventricle consists of the interventricular septum and the free wall, both pericardial and right ventricular pressures should be considered as left ventricular stress boundary conditions. Inclusion of the precise stress boundary conditions, however, must await additional experimental data and three-dimensional finite-element analyses. Here, we consider that the right ventricular pressure and pericardial influence are of the same order of magnitude during diastole, and can be approximated by a single nonzero value of \( P_o \).\(^35\)

Analysis

Based on the above considerations, we examine a finite inflation, extension, torsion, and transmural shearing of a thick-walled, incompressible cylindrical annulus composed of a material described by Equation 1 with muscle fiber orientations defined by the method of solution. The Appendix contains a detailed description of the associated kinematics, constitutive considerations, satisfaction of equilibrium, and method of solution.

General outline. Briefly, we formulate the problem in terms of five deformation quantities: \( r_i \), the inner radius in a loaded, or pressurized, configuration; \( \Lambda \), a uniform axial extension; \( \psi \), a twist per unloaded axial length; and \( \omega(R) \) and \( w(R) \), radially dependent parts of the displacement in the circumferential and axial directions, respectively. The values of \( r_i \), \( \Lambda \), and \( \psi \) are calculated directly from experimental strain data. The radial location of any point in the wall, in a loaded configuration, is then calculated from the incompressibility constraint and knowledge of \( r_i \), \( \Lambda \), and the inner and outer radii in an unloaded configuration, \( R_i \) and \( R_o \). Finally, for the calculation of stress and strain, we only need the derivatives of \( \omega(R) \) and \( w(R) \) with respect to the unloaded radius (i.e., \( \frac{d\omega}{dR} \) denoted as \( \omega' \) and \( \frac{dw}{dR} \) denoted as \( w' \)). We determine \( \omega' \) and \( w' \) from experimental strains at a single location in the wall and from satisfaction of two of the three equilibrium equations. The final equilibrium equation yields that portion of the stress associated with the incompressibility of the tissue.

Preliminary calculations. Using Equations A-23a through A-23e in the Appendix, we calculated representative deformation parameters from the diastolic strain data of Waldman et al.\(^16\) (i.e., the data between 0.4 and 0.57 seconds in Figure 3 and 4 of that study. Waldman et al.\(^16\) calculated these strains from the motions of radially directed tetrahedrons formed by lead markers embedded in the heart wall; thus, their measurements were averaged radially. Consequently, for data from the subendocardial region, we assume that their measurements reflect strains midway through the inner third of the wall (i.e., \( R=1.5 \) cm and \( R_o=2.55 \) cm).\(^30\) With these assumptions and correction for our reference state being the beginning of diastasis (Waldman used end diastole) and our coordinate system being \( R,\Theta,Z \) (Waldman used \( \Theta,Z,R \)), we found that \( r_i=1.77 \) cm, \( \psi=1.77 \) rad/cm, and \( \Lambda=1.015 \). Interestingly, we also calculated \( \Lambda \) from Equation A-3 using the heart dimensions in Olson et al.\(^18\) and obtained \( \Lambda=1.02 \). Finally, we let \( \omega'=0.1 \) rad/cm and \( w'=0.2 \) at \( R=R_o=1.5 \) cm. The remaining values of \( \omega' \) and \( w' \) were determined separately for each radial location by use of the method outlined in the Appendix (Equations A-24 through A-28).

Due to a lack of complete data, we assumed that the material parameters in Equation 1 do not vary across the wall, and employed the following values: \( c=0.115 \) kPa, \( \lambda=9.665 \), \( \Lambda=0.082 \) kPa, and \( \alpha=61.52 \) (where kPa=1,000 N/m\(^2\))\(^1\).\(^7\) Moreover, we let \( P_o=0.98 \) kPa, \( \Phi_o=-90^\circ \), and \( \Phi(R) \) vary cubically (Equation 2). Collectively, these parameters were employed to generate numerical results for an equatorial region in a "representative" canine left ventricle. Finally, we also examined changes in the predicted stresses that were induced by assuming a linear muscle fiber distribution\(^11,12\) instead of a more realistic cubic one (Equation 2); neglecting possible external surface pressures on the heart\(^8,11-13\); neglecting transmural shearing strains\(^8,11-13\); and neglecting possible twisting of the ventricle.

Results

Representative Stresses and Strains

All six components of the Green strain varied monotonically as a function of radius as prescribed by Equations A-1 through A-5 (Figure 2). Because the circumferential (\( E_{cc} \)) and radial (\( E_{rr} \)) strains are influenced primarily by the term \( (r/R)^2 \), their radial gradients near the inner wall can change dramatically (not shown) with even small changes in \( (r/R) \). Thus, there is a need for accurate measurements of ventricular geometry (e.g., Olson et al.\(^18\)) rather than indirect estimates (e.g., Feit\(^8\)). Due to the magnitudes of the deformations, infinitesimal strain assumptions are clearly inappropriate.

In contrast with the transmural distribution of strains, only three of the components of stress varied monotonically as a function of radius (Figure 3). In particular, the radial stress, \( t_r \), varies from \(-P_i \) to \(-P_o \), and the transmural shearing stresses, \( t_z \) and \( t_{\theta} \), are inversely proportional to \( r \) and \( r^2 \), respectively, each as required by equilibrium. Conversely, the circumferential \( (t_{\theta}) \), axial \( (t_a) \), and in-plane shearing \( (t_z) \) stresses had complex transmural distributions. The peaks in \( t_{\theta}, t_a, \) and \( t_z \) in the inner third of the wall result primarily from the anisotropy introduced by the muscle fiber direc-
Figure 2. Representative transmural Green strains are plotted versus undeformed radius, and are based on the following values of the parameters: $R_1=1.5$ cm, $R_o=2.55$ cm, $r_1=1.77$ cm, $\psi=1.7^\circ/cm$, $\omega=0.1$ rad/cm at $R_0$, $\Lambda=1.015$, $w=0.2$ at $R_0$, $\Phi_o=-90^\circ$, $c=0.115$ kPa, $b=9.665$, $A=0.082$ kPa, $a=61.52$, $P_o=0.98$ kPa, and $\Phi(R)$ varies cubically (Equation 2).

As shown in the Appendix, these three components of stress can be used to calculate the stress in the direction of the muscle fibers.

It is noteworthy that these general characteristics of the Cauchy stress hold for any transversely isotropic material defined by a $W(I_1,\alpha)$ with changing muscle fiber directions lying in constant radial surfaces (e.g., $N$ in Equations A-7 and A-8) and, thus, are not restricted to the specific form of $W(I_1,\alpha)$ in Equation 1, or the particular values of the parameters. Therefore, as these figures illustrate, the components of the Cauchy stress cannot be simply inferred from knowledge of the associated strains.

Factors Influencing the Stress Distributions

Figures 4 and 5 illustrate components of the Cauchy stress based on the representative parameters above (Figures 2 and 3), except that we separately assumed a) a linear distribution of muscle fibers, Equation 2 with the \{. . .\} term raised to the first power; b) no external pressure, $P_o=0$; c) no twist, $\psi=0$; d) no RZ shear, $w'=0$; or e) no RO shear, $\omega'=0$. The figures also show the corresponding component of stress from Figure 3 for comparison. As can be seen, the radial stress becomes less negative, at almost all values of $r$, when each of these changes in parameters is invoked (Figure 4A). Thus, a difference in cavity pressure on the order of 15–20% can result, for example, from inclusion or neglect of known amounts\(^{18}\) of transmural shearing strains. Moreover, neglect of an external pressure of 0.98 kPa has a large effect on the magnitude of $t_r$ at all $r$ values, even though the transmural pressure $(P_t-P_o)$ varies only minimally.

The transmural shearing stresses, $t_{\theta\theta}$ and $t_{rr}$, are identically zero if $\omega'=0$ or $w'=0$, and are significantly influenced (on the order of 25%) by the presence or absence of $w'$ and $\omega'$, respectively (Figures 4B and 4C). The presence or absence of a $1.7^\circ/cm$ twist does not significantly affect the distribution of these components of stress, however. Finally, as expected (Equation A-12), alteration of muscle fiber orientations or external pressure does not influence the transmural shearing stresses.

The magnitude of the circumferential stress, $t_{\theta\theta}$, decreases at almost all values of $r$ when the twist and transmural shearing strains are neglected (Figure 5A). Conversely, neglect of $P_o$ increases this component of stress at all values of $r$ (Figure 5D). Marked differences in both the magnitude and character of the stress distribution arise due to linear versus cubic muscle fiber distributions (Figure 5D); that is, the peak value of stress both diminishes and moves toward the midwall by assumption of a linear muscle fiber distribution.\(^{11,12}\) Results for the axial ($t_{zz}$) and in-plane shearing ($t_{\theta z}$) stresses are similar, except that $P_o$ does not affect $t_{\theta z}$ (Figures 5B, 5C, 5E, and 5F). Nonetheless, note the dramatic differences (some on the order of 50%) in $t_{\theta z}$ induced by inclusion or neglect of $P_o$, $w'$, and $\omega'$.
FIGURE 4. Calculated transmural distributions of $t_n$, $t_{th}$, and $t_{zn}$ using parameters noted in Figure 2, except that we separately prescribe either a linear muscle fiber distribution (dotted line), no twist ($\psi=0$, long dashed line), no $R\theta$ shear ($w'=0$, dash-dot line), or no external pressure ($P_0=0$, labeled solid line in panel A). For purposes of comparison, the appropriate component of stress from Figure 3 is shown in each panel as an unmarked solid line. Symbols associated with each curve denote the quantity that has been neglected or changed (e.g., $\Phi$, dotted line, denotes the curve based on a linear, instead of cubic, muscle fiber distribution). Curves are denoted similarly in each panel.

Finally, the stretch ratio in the direction of the muscle fiber (Equation A-7) is illustrated in Figure 6. The effects of twist and the distribution of muscle fibers are pronounced. Although the displacements, $\omega(R)$ and $w(R)$, do not affect $\alpha$, (see also Equation A-9), they clearly affect the stresses, as seen in Figures 3–5.

FIGURE 5. Calculated transmural distributions of $t_{th}$, $t_{zn}$, and $t_{xz}$. Format is the same as in Figure 4 except that the curves are denoted differently in panels A, B, and C ($\psi=0$ is long dashed line, $w'=0$ is short dashed line, and $w'=0$ is dotted line) and panels D, E, and F (no external pressure is dotted line, linear fiber angle is long dashed line).
Discussion

Implications of the Results

Armed with the present or easily generated additional numerical calculations, it is tempting to suggest physiological implications. Although this could be done here, we prefer to resist this temptation and simply note the following: The deformation [i.e., $r/R$, $\psi$, $\omega(R)$, $w(R)$, and $\Lambda$] can influence the stresses in a highly nonlinear fashion. Similarly, different functional forms of the constitutive relations, differing values of the material parameters and boundary conditions, and various descriptions of muscle fiber orientations can markedly change the predicted stresses. Thus, it is difficult, even with the present results, to discern the effects of these parameters categorically. Perhaps most importantly, however, we find that even slight variations in many different parameters can significantly alter not only the magnitudes but also the character of the transmural distributions of stress and strain. For example, it may not seem necessary, or feasible, to experimentally determine the "diastolic twist" in the heart more accurately than $\pm 1^\circ$cm. As shown in Figure 7, however, the circumferential stress, based on the aforementioned representative parameters with $\psi=1.0^\circ$cm or $\psi=3.0^\circ$cm, can differ significantly ($\sim 30\%$) over this small range of twist. In light of these observations, we emphasize that inferences of the distribution of stress in an actual left ventricle must be made judiciously, and be based on accurate experimental data.

Need for Additional Data

Regional variations in myocardial properties have been suggested based on limited data, and on theoretical arguments, but have not been adequately investigated. Since it is easy to show that transmural variations in material properties can significantly affect predicted distributions of stress, there is an obvious need to obtain regional data. Similarly, as pointed out by Fung, it is likely that residual stresses exist in the heart. Residual stresses are those stresses that remain after removal of all external loads. Since residual stresses can significantly influence the actual distribution of stress in organs, experimental and theoretical characterizations of the residual stress phenomena must come forth before we can quantify the true state of stress in the heart.

Accurate experimental data on the deformations experienced by the heart are fundamental to the performance and validation of stress analyses. Hence, reliable in vivo and in situ strain data are of paramount importance. We employed the data of Waldman and colleagues, which appear to be the best data presently available, although likely incomplete. Due to the potentially large radial gradients in strain, particularly in the subendocardium, it is imperative that the appropriate error analyses be performed in the future to ensure confidence in this type of data. Furthermore, the possible presence of residual strains must be explored and documented, and the most appropriate state to which the strains should be referred must be identified.

Finally, a nonzero external pressure can significantly affect the normal stresses. Since the pericardia (visceral and parietal) may contribute substantially to a nonzero value of $P_0$ on the free wall, the mechanics of the pericardia must be better understood. Moreover, the influence of other thoracic...
structures (e.g., great vessel attachments) like that of hilar support on the distribution of stress in the lungs should be investigated.42

Utility of Present Analysis

In spite of the lack of necessary data, theory and experiment must continue to progress in harmony. Our current analysis extends previous attempts to model the heart with a cylindrical geometry in that we present a general solution valid for any constitutive description of myocardium in terms of a transversely isotropic pseudostrain-energy function, W(1, α), including arbitrary distributions of muscle fiber angles that lie in constant radial surfaces. Rather than restricting the myocardial properties to be homogeneous, we can easily include radially dependent material parameters. We also consider a general deformation that is consistent with experimental observations that all six components of the Green strain are significant in the canine heart (i.e., we do not neglect transmural shearing strains as in prior analyses). Moreover, we provide an approach wherein the deformation parameters can be determined directly from experimental strain data and satisfaction of the equilibrium equations. This is in contrast with previous approaches, wherein an incomplete set of deformation parameters is inferred from assumed loading conditions.8,11,12 Until the issues of residual stress and the coupling between systolic and diastolic properties are clarified, these assumed loading conditions will remain questionable.

We also hope that the present analysis, and refinements thereof, can be used by experimentalists to guide their data acquisition. For example, if one desires to predict stresses to within a 10% accuracy given experimental strains, the present analysis can be used to examine possible requirements on the necessary accuracy of the strain measurement (e.g., see Figure 7).

Another utility of our analysis is the ease of inclusion of new information. For example, Chuong and Fung43 suggested that residual strains in arteries can be quantified by consideration of the following "residual deformation." Similar to Equation A-1 in the Appendix, consider material particles in a stress-free configuration (ρ, θ, ξ) to go to (R, Θ, Z) in an unloaded, but residually stressed, configuration, namely

\[ R = R(ρ), \ Θ = (π/Ω_{0})θ, \ Z = αζ \]  

where Ω0 is an opening angle that can be measured when a longitudinal cut is introduced into an artery to relieve residual stresses, and λ is an axial extension ratio that can be measured by observation of the axial shortening or lengthening associated with relief of the residual stresses.

The developments of Chuong and Fung43 can be easily incorporated into our analysis. Thus, if Equation 3, or comparable relations, are shown to adequately describe the residual strains in the equatorial region of the heart, the present analysis can be modified to include residual stresses. The effects of residual stress were not considered here, however, due to a lack of sufficient information.

Finally, it is almost certain that once the requisite experimental data become available, full three-dimensional finite-element analyses will be necessary for the estimation of cardiac stresses. Due to the inherent nonlinearities in cardiac mechanics, however, convergence to the correct solution by finite elements cannot be guaranteed, and thus the predictions must be checked against known solutions. Our analysis, albeit limited to cylindrical geometry, includes many of these nonlinearities and, therefore, could be used to check sophisticated finite-element predictions. Since our analysis may be of use to experimentalists and numerical analysts alike, we will gladly supply interested investigators with our IBM AT FORTRAN programs.

In summary, we reviewed many assumptions commonly employed in the analysis of stress in the passive heart. We concur with Moriarty39 that finite deformations, the finite thickness of the wall, and nonlinear stress-strain relations must be included in any analysis of regional stresses and strains. Additionally, we suggest that a) until sufficient ultrastructural data are available, the continuum hypothesis remains a useful approximation; b) any myocardial constitutive relation must describe available multi-axial data (i.e., isotropic and fiber-reinforced fluid models are not appropriate); c) experimentally measured transmural shearing strains should be accounted for; and d) a cylindrical geometry is perhaps the most useful analytically although it represents, at best, only select regions within the left ventricle.

Consequently, we presented an analysis of a finite deformation of a thick-walled cylindrical annulus composed of a nonlinear, anisotropic, incompressible material. We employed a new pseudoelastic constitutive relation derived from experimental biaxial data on potassium-arrested myocardium, included a reasonable description of the transmural distribution of muscle fibers, and considered finite deformations including inflation, extension, twist, and transmural shearing. Illustrative predictions of transmural variations in stress and strain were presented based on available experimental data. These results are not to be viewed as final, however, due to the lack of complete data.

In contrast with finite-element analyses, our solution has the disadvantage of being restricted to a cylindrical geometry; this limitation does not appear to be severe in the equatorial region of the left ventricle, however. Advantages of our solution over finite elements include the cost- and time-efficient performance of parametric studies, and certain explicit results. For example, the transmural shearing stresses must vary inversely with r or r² as required by equilibrium. Finally, our analysis can serve to check finite-element solutions.
Due to the sensitivity of the stresses to the various parameters, it is essential to base analyses of stresses on complete and reliable data. In particular, there is a need to determine the following: regional stress-strain relations for normal and diseased myocardium in the passive and active states, the properties and contributions of the pericardia, regional strain data throughout the cardiac cycle, the possible effects of trabeculations on endocardial stress concentrations, residual stresses, inertial effects, regional boundary conditions, and so forth. Theory and experiment must proceed hand in hand. Until the requisite data are available, regardless of our computational sophistication, predictions of stress in the heart will remain only speculative and the associated physiological interpretations suspect.

Appendix

Kinematics

We consider finite deformations of a thick-walled, incompressible cylindrical annulus including inflation, extension, twist, and transmural shearing. Thus, we consider material particles in an unloaded configuration \((R, \theta, Z)\) to go to \((r, \phi, z)\) after loading, namely

\[
\begin{align*}
r &= r(R), \\
\theta &= \Theta + \phi Z + \omega(R), \\
z &= Z + w(R)
\end{align*}
\]

where \(\phi\) is a twist-per-unit unloaded length, \(\omega\) is a radially dependent portion of the circumferential displacement, \(\Lambda\) is an extension ratio denoting uniform changes in axial lengths per unloaded lengths, and \(w\) is a radially dependent part of the axial displacement. Analyses of similar deformations of cylindrical tubes are found in Green and Adkins and Truesdell and Noll.

The physical components of the deformation gradient, \(F\), are determined from the motion (Equation A-1), and are conveniently written in matrix form as

\[
F_{mn} = \begin{bmatrix}
\frac{\partial r}{\partial R} & \frac{\partial r}{\partial R\Theta} & \frac{\partial r}{\partial RZ} \\
\frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial R\Theta} & \frac{\partial \theta}{\partial RZ} \\
\frac{\partial z}{\partial R} & \frac{\partial z}{\partial R\Theta} & \frac{\partial z}{\partial RZ}
\end{bmatrix}
\]

\[
\begin{bmatrix}
t' \\
\omega' \\
w'
\end{bmatrix}
\]

wherein the prime denotes an ordinary derivative with respect to the unloaded radius \(R\), as, for example, \(r'\) denotes \(dr/dR\). Incompressibility (\(\text{det } F=1\)) requires that \((r')(\Lambda/r)=1\), which, when integrated, yields either the functional dependence of \(r\) on \(R\), or the overall incompressibility relation

\[
r^2 - r_0^2 = (R^2 - R_0^2)/\Lambda \quad \text{or} \quad r_i^2 - r_0^2 = (R_i^2 - R_0^2)/\Lambda
\]

Subscripts \(i\) and \(o\) denote inner and outer surfaces, respectively.

The physical components of the left and right Cauchy-Green deformation tensors \((B, C)\) are, \(B_{mn} = F_{im}F_{jn}\) and \(C_{mn} = F_{im}F_{jn}\) where \(m, n, M,\) and \(N=1,2,3\). Wherever possible, lowercase and uppercase indices denote quantities associated with the loaded and unloaded configurations, respectively, and repeated indices imply summation, for example, \(trC = C_{mn} = C_{11} + C_{22} + C_{33}\). In matrix form, \(B_{mn}\) is

\[
\begin{pmatrix}
t'(r\omega')^2 \\
(r\omega')^2 \\
(w')^2 + (\Lambda)^2
\end{pmatrix}
\]

and \(C_{mn}\) is

\[
\begin{pmatrix}
t'(r\omega')^2 + (r\omega')^2 + (w')^2 \quad (r\omega')^2 \\
(r\omega')^2 \\
(w')^2 + (\Lambda)^2
\end{pmatrix}
\]

Thus, both \(B\) and \(C\) are symmetric. Waldman et al use the Green strain measure, \(E=(C-1)/2\), which is easily calculated from \(C\).

The first principal strain invariant \((I_1 = trC = trB)\) is

\[
I_1 = t'(r\omega')^2 + (r\omega')^2 + (w')^2 + (\Lambda)^2
\]

Thus, \(I_1\) depends on all of the deformation parameters in Equation A-2 \((r, \phi, \Lambda, \omega, \) and \(w')\). It will also prove useful to calculate the stretch ratio, \(\alpha\), in the direction of a muscle fiber. Thus, we consider

\[
\alpha^2 = N\cdot C\cdot N = C_{MN}N_MN_N
\]

where \(N\) is a unit vector in the direction of a muscle fiber in the unloaded configuration. For example, we can assume that \(N\) has components

\[
[N_1, N_2, N_3] = [0, \cos \Phi(R), \sin \Phi(R)]
\]

where \(\Phi(R)\) describes the transmural distribution of muscle fibers through the heart wall (e.g., Equation 2 in the text, although other descriptions may be physiologically more realistic). Regardless, \(\alpha^2\) is

\[
\alpha^2 = [(r/R)^2 \cos^2 \Phi(R) + 2(r/R)\sin \Phi(R) \cos \Phi(R)]
\]

+ \([r(\psi)^2 + (\Lambda)^2]\) \sin^2 \Phi(R)

Thus, the stretch ratio in the direction of a muscle fiber is independent of \(w(R)\) and \(\alpha(R)\), but depends on \(\Phi(R), r, \psi, \) and \(\Lambda\).

Constitutive Relation

Calculation of stresses from the deformation requires knowledge of a constitutive (stress-strain) relation for the tissue. For the present purposes, we consider noncontracting myocardium to be transversely isotropic with respect to the directions of the muscle fibers; that is, we assume that myocardium responds equally to loads applied in any direction excluding the muscle fiber direction wherein a different force-deformation behavior exists. A general form of a transversely isotropic pseudostrain-energy function for myocardium can be written as a function of \(I_1\) and \(\alpha^2\)

\[
W = W(I_1, \alpha^2)
\]
A specific form of $W(I, \alpha)$ is provided in the text (Equation 1). The Cauchy stress (force/current area) for a material described by Equation A-10 is given by

$$t = -pI + 2W_1 B + \frac{W_a}{a} F - N g$$  \hspace{1cm} (A-11)

where $t$ is the Cauchy stress, $p$ is a Lagrange multiplier enforcing incompressibility, $I$ is the identity tensor, $W_i = \partial W / \partial I_i$, $W_a = \partial W / \partial \alpha$, $\otimes$ denotes tensor product, and the superscript $T$ denotes transpose. Thus, from Equations A-2, A-4, A-10, and A-11, the nonzero physical components of the Cauchy stress are

$$t_{rr} = -p(r) + 2W_1 \left( r' \right) \left( r' \right)^2$$ \hspace{1cm} (A-12a)

$$t_{rz} = -p(r) + 2W_1 \left[ (r/R)^2 + (r' \omega)^2 + (r/a)^2 \right] + \left( W_a / a \right) \left[ a^2 - \lambda^2 \sin^2 \Phi(R) \right]$$ \hspace{1cm} (A-12b)

$$t_{r\phi} = 2W_1 \left( r' \omega' \right)$$ \hspace{1cm} (A-12c)

where $t = t^*; \Phi(R)$ can be any transmural distribution of muscle fibers, and similarly $W_i$ and $W_a$ can be calculated for any constitutive relation expressed by $W(I, \alpha)$. For example, for Equation 1 in the text

$$W_1 = bce^{(n-1)}$$

and $W_a = 2a(\alpha - 1)Ad^{(n-1)^2}$

Since $W_i$ and $W_a$ may depend on $I_i$, and since $I_i$ depends on $\omega(R)$ and $w(R)$, each of the components of stress can depend on $w$ and $\omega$, components of the deformation previously neglected in cylindrical models of the heart. Finally, the components of stress and strain are a function of only one spatial variable, $r$, thereby keeping the analysis tractable.

**Stress Transformations**

The component of Cauchy stress in the direction of the muscle fibers is easily determined from the usual transformation relations ($t_{mn}^* = Q_{mn}Q_{np}$, where $Q_{mn}$ are the directions cosines for the transformation and the * denotes a transformed quantity). Thus, the stress in the direction of a muscle fiber is

$$t_{zz}^* = t_{zz} \cos^2 \phi(r) + 2t_{rz} \cos \phi(r) \sin \phi(r) + t_{r\phi} \sin^2 \phi(r)$$ \hspace{1cm} (A-12g)

wherein the deformed muscle fiber angle ($\phi$) is

$$\phi(r) = \tan^{-1} \left( n_z / n_r \right)$$ \hspace{1cm} (A-13)

with $n = F/N/\alpha$ and $n_i = [0, \cos \phi(r), \sin \phi(r)]$. Thus, the stress in the direction of a muscle fiber depends only on three components of the Cauchy stress: circumferential, axial, and in-plane shear. Finally, remember that the stress in the muscle fiber direction is a continuum quantity reflecting locally averaged loads borne by muscle, collagen, and other constituents.

**Equilibrium and Boundary Conditions**

In the absence of body forces, the equilibrium equations ($\text{div} \ t = 0$) are

$$\frac{dt_{rr}}{dr} + \frac{t_{r\phi}}{r} = 0$$ \hspace{1cm} (A-14a)

$$\frac{dt_{r\phi}}{dr} + 2t_{rz}/r = 0$$ \hspace{1cm} (A-14b)

$$\frac{dt_{rr}}{dr} + t_{r\phi}/r = 0$$ \hspace{1cm} (A-14c)

which are easily integrated, uncoupled ordinary differential equations. For example, Equation A-14b can be written as

$$\frac{d(r^2t_{rr})}{dr}/r^2 = 0$$ \hspace{1cm} (A-15a)

from which we obtain

$$r^2 t_{rr} = \text{constant} = H$$ \hspace{1cm} (A-15b)

Similarly, Equation A-14c can be written as

$$d(r_t)/dr = 0$$ \hspace{1cm} (A-16a)

and thus

$$rt_{\phi} = \text{constant} = G$$ \hspace{1cm} (A-16b)

Equilibrium requires, therefore, that $t_x$ and $t_y$ are inversely proportional to $r^2$ and $r$, respectively, both independent of the particular form of $W(I, \alpha)$.

If we assume that the inner and outer surfaces of the cylinder experience uniform pressures (Figure 1C), then the radial stress boundary conditions are

$$t_{rr}(r) = -P_1 \text{ and } t_{r\phi}(r) = -P_o$$ \hspace{1cm} (A-17)

Integration of Equation A-14a, with the aid of Equation A-17, yields the expression for the "transmural pressure" $P$:

$$P = P_0 + \int_{r_i}^{r_o} \{r_{r\phi} - \text{constant}\} dr$$ \hspace{1cm} (A-18)

or the Lagrange multiplier in Equations A-11 and A-12

$$\frac{dP}{dr} = 2W_1 \left( r' \right)^2 + P_0 - \int_{r_i}^{r_o} \{r_{r\phi} - \text{constant}\} dr$$ \hspace{1cm} (A-19)

where the integrands are

$$\{r_{r\phi} - \text{constant}\} = \left( W_1 \right) \frac{(r/R)^2 + (r\omega)^2 + (r/a)^2 - (r')^2}{r} + \left( W_a / a \right) \left[ a^2 - \lambda^2 \sin^2 \Phi(R) \right]$$ \hspace{1cm} (A-20)

These equations are easily integrated numerically via a Romberg method. The correctness of our numerical algorithms was verified by comparison of the solution of a finite inflation of a thick-walled Mooney-Rivlin cylinder with a finite-element solution by a commercial program (ABAQUS).

If one wishes to calculate the externally applied loads necessary to maintain this deformation (Equation A-1), then one might consider, for example, the axial load ($L$) and moment ($M$) equations

$$L = 2\pi \int_{r_i}^{r_o} (t_{rr} + P_1 \pi r_i + P_o \pi r)^2$$ \hspace{1cm} (A-21a)
Equations A-21a and A-21b were used in three studies\textsuperscript{8,11,12} to infer values of the deformation parameters \((A, \psi)\) by the assumption that \(L=0\) and \(M=0\) were balanced by the appropriate integrals over passive stresses. If \(L\) and \(M\) are actually zero, however, it is likely that they will be balanced by the total stress \(\sigma\) and \(\tau\), stresses, including contributions from residual, passive, and perhaps active (associated with diastolic tone) stresses. Thus, unless the total state of stress is used, deformation parameters inferred from assumed gross balance relations are suspect. Since the total stress state is not known at present, we prefer to calculate the values of the deformation parameters directly from experimental strain data.

**Method of Solution**

Components of the Green strain, \(E\), are obtained from Equation A-4 and are

\[
E_{rr} = \frac{(r')^2 + (\tau'\rho')^2 + (w')^2 - 1}{2} \quad (A-22a)
\]

\[
E_{\phi\phi} = \frac{(\tau/R)^2 - 1}{2} \quad (A-22b)
\]

\[
E_{zz} = \frac{(\rho\psi)^2 + (\Lambda)^2 - 1}{2} \quad (A-22c)
\]

\[
E_{\rho\phi} = \frac{(\tau\rho')/(r/R)}{2} \quad (A-22d)
\]

\[
E_{\rho\pi} = \frac{(\tau\phi')(\psi')}{2} \quad (A-22f)
\]

Thus, if the components of the Green strain are known at a single radial location in the wall of the heart, \(R\), then we can calculate the deformation parameters \((r, \psi, \Lambda, \omega', w')\) in Equation A-2 directly. That is, from Equations A-22b, A-22f, A-22c, A-22d, and A-22e, respectively, we have

\[
r = R\sqrt{1 + 2E_{\phi\phi}} \quad (A-23a)
\]

\[
\psi = (R/r')(2E_{\rho\phi}) \quad (A-23b)
\]

\[
\Lambda = \sqrt{1 + 2E_{zz} - (\rho\psi)^2} \quad (A-23c)
\]

\[
\omega' = \frac{(2E_{\rho\phi} - (\tau\rho')(\psi')/\Lambda}{E_{\rho\phi}} \quad (A-23d)
\]

\[
w' = \frac{(2E_{\rho\phi} - (\tau\rho')(\psi')/\Lambda}{E_{\rho\phi}} \quad (A-23e)
\]

Equation A-22a can be used to check the assumption of incompressibility.

Since the values of \(\psi\) and \(\Lambda\) in Equation A-1 do not vary with radial location, Equations A-23b and A-23c yield values that are valid at all values of \(R\). Similarly, \(r\) is easily evaluated at any radial location by use of the incompressibility constraint \(r' = (r/R)\Lambda\). Conversely, \(\omega'\) and \(w'\) vary with radial location, and although they are known at one point in the wall via Equations A-23d and A-23e, they are yet undetermined functions of radius. We suggest one possible approach to determine \(w'(R)\) and \(\omega'(R)\).

Once we know the deformation parameters at a single radial location (Equations A-23a through A-23e), we can then determine the integration constants, \(G\) and \(H\), from Equations A-12, A-15, and A-16 as

\[
G = [2W_1(r'w')']_R \quad (A-24a)
\]

and

\[
H = [2W_1(r'w'\rho')']_R \quad (A-24b)
\]

where \(\lvert\cdot\rvert\) denotes "evaluated at" the \(R\) value corresponding to the strains used in Equation A-24. With \(G\) and \(H\) known, Equations A-24a and A-24b can be used to solve for all values of \(\omega'\) and \(w'\).

Although \(a\) is independent of \(w'\) and \(\omega'\), \(I_1\) depends nonlinearly on both \(w'\) and \(\omega'\). Thus, unless \(W_1\) is independent of \(I_1\) (e.g., Neo-Hookean rubber where \(W = c[I_1 - 3]\) and \(c\) is a material parameter), Equation A-24 represents two coupled nonlinear algebraic equations for \(w'\) and \(\omega'\) (when both are nonidentically equal to zero). These equations are decoupled and solved easily, however, by consideration of the following ratio of transmural shearing stresses

\[
t_\omega = \frac{t_\omega}{t_\tau} = \frac{(\rho\psi')}{(w')}
\]

which is valid for all \(W(I_1, a)\). We find \(\omega'\) in terms of \(w'\), therefore, as

\[
\omega' = (H/G)t_\omega
\]

Substituting \((H/G)t_\omega\) into \(I_1\) (Equation A-6) for \(\omega'\) and letting \(x\) denote \(w'\), we obtain

\[
I_1 = (r')^2 + (r/R)^2 + (\psi')^2 + (\Lambda)^2 + 1 + (H/G)t_\omega\exp^2 \quad (A-27)
\]

Thus, Equation A-24a becomes an uncoupled nonlinear algebraic equation in terms of \(w'\) (or \(x\)), and can be written as a function of \(x\)

\[
f(x) = 2W_1(r'(\psi' + x) - G = 0 \quad (A-28a)
\]

Moreover,

\[
df/dx = 2(r'(\psi' + x)W_1 + x(dW_1/dx))\]

and, thus, we can determine \(x\) (or \(w'\)) via a straightforward root-finding algorithm. We chose the Newton-Raphson method\textsuperscript{49} to determine values of \(x\) by iteratively improving on an initial guess for \(x\) (say \(x_0\)), namely

\[
x_{n+1} = x_n - \frac{f(x_n)}{df/dx} \quad (A-28c)
\]

where \(x_n\) is the previous value (\(n = 1\) is the initial guess) and \(x_{n+1}\) is the current value (\(n = 1, 2, \ldots\)). Iterations continue until the current value of \(x_{n+1}\) converges to within the desired accuracy. This Newton-Raphson method works well when reasonable initial "guesses" for \(x\) are available. Here, we can suggest judicious initial guesses for \(x\) since we know one value of \(x\) from the strain data (Equation A-23e). Note, too, that for Equation 1 in the text

\[
dW_1/dx = 2(xb[1 + (H/G)t_\omega])W_1
\]

After \(w'\) is known at each radial location, we determine \(\omega'\) from Equation A-26. If either \(\omega'\) or \(w'\) is identically zero at all radial locations, however, then either \(G\) or \(H\) is zero, Equations A-24a and

\[\text{Humphrey and Yin} \quad \text{Stress Analysis in Passive Myocardium} \quad 815\]
A-24b become uncoupled, and the nontrivial equation can be solved directly via a Newton-Raphson method.

Given \( w' \) and \( \omega' \) at all values of \( r \), one can then solve Equations A-18 and A-19 and thus compute the stresses and strains at any radial location. Hence, this approach yields "analytical" solutions, except for the numerical integrations and root finding for \( w' \) and \( \omega' \). Finally, \( w(R) \) and \( \omega(R) \) are difficult to determine directly since the necessary equations are coupled and nonlinear, but can be reconstructed from knowledge of \( w'(r) \) and \( \omega'(r) \). Only the derivatives of \( w(R) \) and \( \omega(R) \) are needed for calculation of stress and strain, however, and there is no need to reconstruct the actual displacements.

References

8. Feit TS: Diastolic pressure-volume relations and distribution of pressure and fiber extension across the wall of a model left ventricle. *Biophys J* 1979;28:143-166

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