The Law of Laplace

Its Limitations as a Relation for Diastolic Pressure, Volume, or Wall Stress of the Left Ventricle

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SUMMARY I made a detailed comparative examination of five mathematical models of left ventricular (LV) mechanics: the Laplace model, Lamé model, Valanis-Landel model, Rivlin-Saunders model, and a nonhomogeneous version of the Valanis-Landel model. All five models are used to predict LV pressure-volume (P-V) and pressure-wall stress (P-S) behavior using the same geometric and stress-strain data (rat data is used as an example). These predictions are presented in graphical form for comparison with each other and observed LV P-V behavior. The first model, based on the Law of Laplace, uses the thin wall approximation in three distinct ways, and this approximation is not consistent with LV mechanics. The small deformation and linear stress-strain assumptions of the Lamé model also are inconsistent with LV mechanics. Two more homogeneous models (Valanis-Landel and Rivlin-Saunders) avoid the errors of the first two. The discrepancy in the predictions of these two models demonstrates that uniaxial stress-strain data of papillary muscle are insufficient to characterize the multiaxial stress-strain behavior of myocardial tissue. Finally, a nonhomogeneous version of the Valanis-Landel model which explores the variation of myocardial stress-strain behavior needed to achieve constant wall stress is presented. Circ Res 46: 321-331, 1980

AN ACCURATE determination of the interrelationship of pressure, volume, and wall stress of the left ventricle (LV) is an important prerequisite for a fundamental understanding of cardiac mechanics. This interrelationship is a mathematical model which is used to predict wall stress from observed pressure and geometry (Ford, 1976; Rackley, 1976; Mirsky, 1974; Spotnitz and Sonnenblick, 1973; Streeter and Hanna, 1973a, Gould et al., 1972; Falsetti, et al., 1970, 1971; Burns et al., 1971) and stress-strain relations for myocardial tissue from either LV pressure-volume behavior (Glantz, 1976; Rackley, 1976; Fester and Samet, 1974; Mirsky and Parmley, 1973; Lafferty et al., 1972; Diamond et al., 1971), or stress-strain relations of papillary muscle. It also is used to assess myocardial contractility (Falsetti et al., 1971; Hugenholz et al., 1970) and predict LV volume from pressure (Burns et al., 1971) or vice versa (Sonnenblick and Strobeck, 1977). A mathematical model of LV mechanics is essential for the unified understanding of LV morphology in abnormal and diseased states. It may be based on the Law of Laplace (Woods, 1892), the work of Lamé (1866), or the work of others, but whatever the source of the mathematical model, it is important that the assumptions on which it is based are satisfied for the range of deformation and stress-strain behavior exhibited by the LV; otherwise it will yield inaccurate predictions of LV behavior, confounding the observations rather than unifying them.

In this paper, five mathematical models of the mechanics of the relaxed LV are presented, including the equations for the LV pressure-volume (P-V) relations and LV pressure-wall stress (P-S) relations predicted by these models. These equations are general in the sense that specific values for geometry and material properties are not used. The LV P-V and P-S relations predicted by these models are presented in graphical form for comparison with each other, and, in the case of the LV P-V relation, for comparison with test results. For the graphical presentation, the same data on geometry and material properties are used (rat data are used as an example). The first model is based on the law of Laplace and the second on the law of Lamé. The last three models are based on more recent work.

Analysis

The geometry of the LV plays a major role in determining the complexity of the mathematical model needed to represent LV mechanics. To keep the mathematical models as simple as possible, so as not to obscure the points to be made, the LV will be represented as a thick hollow sphere of uniform wall thickness with isotropic, homogeneous (except for the last model), material properties. Since more complex LV models with nonspherical geometry, varying wall thickness, anisotropic and nonhomogeneous material properties must include these simple spherical models as special cases, the findings of this paper are germane to all LV models no matter how complex.
The values taken for the endocardial and epicardial radii (0.25 cm and 0.50 cm, respectively) are taken from Janz and Grimm (1972) as typical values from serial sections of a potassium-arrested adult Sprague-Dawley albino male rat LV. Two spherical coordinate systems will be used throughout this paper: (R,Θ,Φ) for the undeformed (stress-free) state, and (r,θ,φ) for the deformed state. The coordinates in each system are the radius, longitude, and co-latitude, respectively.

Stress-strain relations for myocardial tissue currently are synthesized by using stress-strain results obtained from uniaxial tests of LV papillary muscle together with information on the structure of the myocardium (e.g. Bergel and Hunter, 1979). For materials that deform to as large an extent as papillary muscle and the myocardium, it is convenient to use stretch, \( \lambda \), as a measure of material deformation rather than engineering strain, \( e \). The stretch of a material at a given point and in a given direction is defined as the ratio of the length of a line segment at that point in that direction in the deformed state to the length of the same line segment in the undeformed (stress-free) state. Engineering strain of a material at a given point and in a given direction is calculated by projecting the increased length of the line segment caused by the deformation onto the original direction of the line segment; the length of this projection is then divided by the original length of the line segment (see Timoshenko and Goodier, 1951, pages 5 and 6). For deformations that do not change the direction of the line segments of interest, such as those considered in this paper, engineering strain is equal to the change of length of the line segment divided by its original length, and the relation between stretch and engineering strain for these deformations is:

\[
\lambda = 1 + e \tag{1}
\]

However, for deformations where line segments of interest are rotated by the deformation (as are encountered in detailed LV models of nonspherical geometry), the relationship between stretch and engineering strain is more complex than Equation 1, and engineering strain becomes a less meaningful direct measure of deformation. For this reason stretch is preferred over engineering strain by most investigators concerned with the mechanics of finite deformations of nonlinear stress-"strain" material.

Stress (true stress, Cauchy stress), \( S \), is defined as the ratio of force to the actual (deformed) area on which the force acts. The stress-stretch data for relaxed rat papillary muscle in uniaxial tension tests (Grimm et al., 1970), shown in Figure 1, are taken from Janz and Grimm (1973, Fig. 1) where the undeformed length of the papillary muscle is set at 80% of the length at maximum developed tension. For physiological pressures the papillary muscle is essentially incompressible; consequently the volume of the papillary muscle remains constant during its deformation. For uniaxial papillary muscle tests, if \( \lambda \) is the stretch along the axis of the specimen and \( L \) is the stretch transverse to this axis, then the incompressibility condition can be written:

\[
\lambda L^2 = 1. \tag{2}
\]

If \( A \) and \( a \) are the undeformed and deformed papillary muscle cross sectional areas, respectively, then by Equation 2:

\[
a = \lambda L^2 = A/\lambda. \tag{3}
\]

If \( F \) is the force exerted by the papillary muscle, then by Equation 3:

\[
S = F/a = FA/A. \tag{4}
\]

The stress (true stress, Cauchy stress), \( S \), of Figure 1 was determined from Janz and Grimm (1973, Fig. 1) with the developed tension set equal to 2.5 g at the optimum length (stretched length of papillary muscle at which developed tension is maximum); and \( A = 1.0 \) mm$^2$ (Grimm et al., 1970).

The loading used for the model predictions presented in graphical form is: 0.0 g/cm$^2$ pressure on the epicardium, and 2.5 to 25.0 g/cm$^2$ pressure on the endocardium (the normal LV diastolic pressure range for the rat is 2.5 to 12.0 g/cm$^2$).
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P = 3.0 g/cm²

Normalized Volume, V/V₀

Radial Position, R (cm)

Normalized Volume, V/V₀

Radial Position, R (cm)

Figure 2 A: Predicted LV pressure-volume relations; B, C: predicted LV wall stress distributions for diastolic LV pressures of 3.0 and 6.0 g/cm². Curve a represents test results (Janz and Grimm 1972, 1973) (not available for wall stress distributions); curves b through f are predicted by the Laplace, Lamé, Valanis-Landel, Rivlin-Saunders, and nonhomogeneous Valanis-Landel models, respectively.

The predicted P-S relations will be compared with each other; reliable test data of wall stress distribution is not yet available. The predicted P-V relations will be compared with a P-V relation, Figure 2A (curve a) which was obtained from the observed geometry of potassium-arrested LV's of the rat at four pressure levels (Janz and Grimm, 1972, Figs. 5-8), using the procedure discussed by Janz and Grimm (1973, Appendix II) wherein the meridional curve of the endocardium is approximated as a 12-vertex polygon and the endocardium modeled as a surface of revolution about the apex-valve axis.

Although the relevance of these P-V data, which were obtained from potassium-arrested hearts, to the behavior of a live heart in diastole can be questioned, there are no other reliable data for the rat P-V relation. In any event, these data are used for illustrative purposes only. This qualification should be kept in mind when the phrase “observed P-V relation” is used below.

Laplace Model

For a thin spherical shell with deformed inner radius r, deformed thickness t, that is subjected to internal pressure P, the Law of Laplace is:

\[ S = \frac{Pr}{2t} \]  

where S is the true wall stress, and in deriving Equation 5 use has been made of the hypothesis that the shell is thin, i.e.:

\[ t \ll r. \]

This relationship implies that:

\[ T \ll R \]

where T and R are the undeformed thickness and inner radius, respectively. To determine the LV P-V and P-S relations with the Law of Laplace, the stress-stretch relation for the myocardium must be introduced. If one uses the Valanis-Landel stress-stretch relation (which is discussed in the development of the Valanis-Landel Model below), the relationship between wall stress, S, and wall stretch, \( \lambda \), is:

\[ S = C[\lambda^\alpha - 1/\lambda^\alpha] \]

where C and \( \alpha \) are material constants (equal to 2.37 g/cm² and 18, respectively, for rat papillary muscle) (see Fig. 1, curve a). From the incompressibility assumption the volume of the LV wall must be constant, therefore:

\[ (R + T)^3 - R^3 = (r + t)^3 - r^3. \]

If the geometry of the LV were such that the thinness conditions 6 and 7 were satisfied, then the incompressibility condition 9 could be approximated accurately by:

\[ R^2T \approx r^2t \]

and the wall stretch, \( \lambda \), could be approximated accurately by the wall stretch at the endocardium:

\[ \lambda = r/R. \]

By using Equations 10 and 11, the Law of Laplace becomes:

\[ S = \frac{(PR/(2T))^{3/2}}{R}. \]

Subtracting the wall stress of expression 8 from the wall stress of expression 12 results in a root finder problem of the form:

\[ F(\lambda) = 0 \]

where the value of \( \lambda \) which satisfies Equation 13 must be found for the given values of R, T, and P.
Once the $\lambda$’s are known for a range of diastolic pressures, the LV P-V relation is known, since, by Equation 11:

$$\frac{V}{V_0} = \lambda^3$$  \hspace{1cm} (14)

where $V$ and $V_0$ are the deformed and undeformed LV volumes, respectively. The LV P-S relation is readily evaluated using either Equation 8 or 12. The LV P-V and P-S relations of the rat predicted by this Laplace model are shown in Figures 2, A–C, and 3, A–C, curves b.

**Lame Model**

Lamé (1866) solved the problem of a thick hollow sphere subjected to internal or external pressure. His solution, however, is based on the assumptions that the sphere is only infinitesimally deformed and that the stress-strain relation for the material is linear. The results of the Lamé model are presented here for comparison with the more exact models. For a detailed discussion of the development of the Lamé model see Timoshenko and Goodier (1951).

To use Lamé’s results, the value for Young’s modulus, $E$, which appears in the linear engineering stress-strain equation (Eq. 15), must be determined for papillary muscle.

$$\sigma = E \epsilon$$  \hspace{1cm} (15)

where engineering stress, $\sigma$, is defined as force divided by the undeformed area to which the force is applied. The stress-stretch data of Figure 1 are converted to an engineering stress-strain plot of Figure 4 by using the definition of engineering stress along with Equations 1 and 4. Figure 4 clearly shows that the relationship between engineering stress and strain is not linear over the full range of the test data, but, for small strains, the data may be approximated by a linear stress-strain relation with a Young’s modulus of 105 g/cm$^2$.

By the spherical symmetry of all the models presented in this paper, the meridional and circumferential components of stress in the LV wall ($S_m$ and $S_c$, respectively) are equal to each other. In the Laplace (thin wall) model presented above, the term “wall stress” was used without ambiguity, since that model ignores the presence of a radial stress component ($S_r$). In this and the following models, which take into account the presence of radial stress, the term “wall stress” will refer to the meridional/circumferential components of stress only.

The LV P-V and P-S relations predicted by the Lamé model are:

$$\frac{V}{V_0} = \left[ 1 + \frac{3PB^3}{4E(B^3 - A^3)} \right]^3$$  \hspace{1cm} (16)

$$S = \frac{\sigma}{P} \frac{A^3(2R^3 + B^3)}{[2R^3(B^3 - A^3)]}$$  \hspace{1cm} (17)

where $A$ and $B$ are the undeformed radii of the endocardium and epicardium, respectively. (See

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**Figure 3**  A–C: Predicted LV wall stress distributions for diastolic LV pressures of 12.0, 18.0, and 24.0 g/cm$^2$. Curves b through f are predicted by the Laplace, Lamé, Valanis-Landel, Rivlin-Saunders, and the nonhomogeneous Valanis-Landel models, respectively.
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Figures 2, A-C, and 3, A-C, curves c.) In Equation 17 the true wall stress, $\sigma$, is approximated by the engineering wall stress, $\sigma_e$, because Lamé's model is valid only when the deformation is very small; and, for very small deformations, engineering stress closely approximates true stress.

Valanis-Landel Model

This is the first of the LV models presented here that simultaneously treats the effects of finite deformation of the LV, the nonlinear stress-stretch behavior of papillary muscle, and the thickness of the LV wall.

The mathematical model of a large spherically symmetric expansion of a thick wall sphere made of an incompressible, isotropic, nonlinear elastic material was treated by Green and Shield (1950) and Green and Zerna (1954). To obtain predictions from the model, however, specific constitutive relations must be used. In this model the following stress-stretch relation, which is a member of a family of relations proposed for hyperelastic materials (i.e., elastic materials for which the work done in elastic deformation is stored as internal energy—for most if not all practical purposes, the terms elastic and hyperelastic are synonymous) by Valanis and Landel (1967), will be used:

$$ S_i = C \lambda_i^n + H $$  \hspace{1cm} (18)

where: $S_i$ is the $i$th principal stress, $\lambda_i$ is the $i$th principal stretch, $C$, $\alpha$ are material constants, and $H$ is a hydrostatic pressure term.

At every point in a body of an isotropic material there are three mutually perpendicular directions in which the stresses and stretches reach a maximum, a minimum, and an intermediate value relative to values in all other possible directions at that point; these stresses and stretches are called principal stresses and stretches. By the spherical symmetry of the LV models, the principal directions lie in the radial, circumferential, and meridional directions, whereas for the uniaxial tests the principal directions lie along the axis of the specimen and perpendicular to this axis.

The hydrostatic pressure term, $H$, in Equation 18 is so named because it represents a pressure that is the same in all directions—a condition which prevails for pressures in fluids in general and water in particular. Static refers to the fact that this term is only a function of position and is not a function of time. This term, hydrostatic pressure, is used in contradistinction to the usual state of pressure in solids where pressure changes with direction, as is the case for solids that can be characterized by Equation 18. For compressible materials, stress can be completely determined from the deformation (stretch) and vice versa. Incompressible materials, however, are not deformed by the application of pressures that are equal in all directions. For such materials the stress is the sum of a term determined by the deformation and a term which adds the stress not reflected by the deformation (a hydrostatic pressure term). (In a mathematical approach not used in this paper, $H$ plays the role of a Lagrange multiplier associated with the constraint on the deformation of the material that the deformation be such that the volume of the material remains constant.)

In the Discussion, below, the connection between this particular Valanis-Landel stress-stretch relation, Equation 18, and some popular exponential stress-strain relations will be brought out.

For the uniaxial test, let $S$ and $\lambda$ be the stress and stretch along the test axis; then, for this principal direction, Equation 18 becomes:

$$ S = C \lambda^n + H $$  \hspace{1cm} (19)

For the two principal directions perpendicular to the test axis, the principal stresses are zero and the principal stretches are each equal to $\lambda^{1/2}$ by the incompressibility condition (Eq. 2). For these two principal directions, Equation 18 becomes:

$$ 0 = C/\lambda^{n/2} + H $$  \hspace{1cm} (20)

By eliminating the unknown hydrostatic pressure term in Equations 19 and 20, the uniaxial stress-stretch relation for the Valanis-Landel material becomes:

$$ S = C(\lambda^n - 1/\lambda^{n/2}) $$  \hspace{1cm} (21)

[To determine the material constants $C$ and $a$ for rat papillary muscle, the uniaxial test data of Figure 1 were normalized by dividing the stress at any given $\lambda$ by the stress at $\lambda = 1.25$. (The average value of stress at $\lambda = 1.25$ was 131 g/cm$^2$.) The uniaxial...}
Valanis-Landel relation 21 was normalized in the same way. Both the normalized test data and the normalized Valanis-Landel relation for various values of $a$ were plotted, and it was found that $a = 18$. Setting, $S$, $A$, and $a$ equal to 131 g/cm$^2$, 1.25, and 18, respectively, the elastic modulus, $C$, was found to be equal to 2.37 g/cm$^2$. For a comparison of the Valanis-Landel relation with test data, see Figure 1, curve a.

The relations between the principal stretches and the deformed and undeformed coordinates are:

$$\lambda_r = \frac{dr}{dR}$$

and

$$\lambda_s = \lambda_a = r/R.$$  

Incompressibility requires:

$$\lambda_r \lambda_s \lambda_a = 1.$$  

Combining Equations 22 through 24 results in a differential equation for $r$ in terms of $R$ whose solution is:

$$r^3 = R^3 + c$$  

where $c$ is a constant of integration which is determined by the boundary condition discussed below. Substituting Equation 25 into Equations 22 and 23 leads to:

$$\lambda_r = [R^3/(R^3 + c)]^{2/3}$$  

and

$$\lambda_s = [(R^3 + c)/R^3]^{1/3}.$$  

The force equilibrium condition is:

$$\frac{d(S_r)}{dr} + \frac{2}{r} (S_r - S_s) = 0.$$  

The two boundary conditions that must be satisfied are:

$$S_r = -P @ R = A \text{ (the undeformed radius of the endocardium)}$$

and

$$S_r = 0 @ R = B \text{ (the undeformed radius of the epicardium).}$$

This model is completed by substituting Equations 18, 26, and 27 into the equilibrium Equation 28 and integrating once to obtain the hydrostatic pressure term $H$. With $H$ now known it is substituted into the expression for $S_r$, Equation 18, and that expression made to satisfy boundary condition 29. With this done, the expression for radial stress becomes:

$$S_r = 2C \int_A^R R^2(U - 1/U^3)/(R^3 + c) \, dr - P$$  

where

$$U = [(R^3 + c)/R^3]^{2/3}$$  

and $R$ is the particular undeformed radius at which the value of radial stress is desired. Note that at the endocardium, where $R = A$, the predicted radial stress is equal to minus the LV pressure, as required.

One unknown remains in Equation 31, namely $c$, which was introduced in Equation 25. This last unknown is determined by requiring the expression for radial stress, Equation 31, to satisfy boundary condition 30. With $R$ and $S_r$ set to $B$ and zero, respectively, Equation 31 becomes a root finder problem for the unknown, $c$. Once a set of $c$’s is determined which spans the desired range of diastolic pressure, the LV P-V and P-S relations predicted by the Valanis-Landel model can be evaluated readily. These relations are:

$$V/V_0 = 1 + c/A^3$$

and

$$S = S_r + C(U - 1/U^2)$$

where $U$ is defined in Equation 32. See Figures 2, A-C and 3, A-C, curves d, for the predicted behavior for the rat.

**Rivlin-Saunders Model**

This model is the same as the Valanis-Landel model, with the sole exception that the Valanis-Landel stress-stretch relation, Equation 18, is replaced by one that is a member of a family proposed by Rivlin and Saunders (1951):

$$S_r = 2C/A^4 + 12D(A^3 - 3/A + 2/A^3) + H$$

and

$$S = 2CA^2 + 6D(A^2 + 1/A^4)(AS - 3A + 2/A)^1 + H$$

where $C$ and $D$ are material constants, $H$ is hydrostatic pressure, and $S$ and $A$ are wall stress and wall stretch, respectively.

For this Rivlin-Saunders material, the uniaxial stress-stretch relation is:

$$S = CV + DW$$

where:

$$V = 2(\lambda^2 - 1/\lambda)$$

and

$$W = 6(\lambda - 1/\lambda^2)(2A + 1/A^2 - 3)^2.$$  

[The values of $C$ and $D$, 21.13 g/cm$^2$ and 1429.46 g/cm$^2$, respectively, of Equation 37, were determined for rat papillary muscle by multiple regression (Coxton, 1959) using the test data of Figure 1. For a comparison of the Rivlin-Saunders uniaxial stress-stretch relation with the test data, see Figure 1, curve b.)]

This model is completed in the same fashion as is the Valanis-Landel model. For this model, the
The general expression for radial stress is:

$$S_r = 4cC \int_A^R (Q^7 + Q^4)/R^4 \, dR$$

$$+ 12cD \int_A^R (Q^6 + Q^4)(2Q^2 - 3) \, dR - P$$

$$+ 1/Q^3 \int_A^R dR - P$$

(40)

where

$$Q = [R^7/(R^3 + c)]^{1/3}$$

(41)

The LV P-V relation is Equation 33. The LV P-S relation is:

$$S = S_r - 2C(Q^4 - 1/Q^2)$$

$$- 6D(Q^2 - 1/Q^4)(2Q^2 - 3 + 1/Q^4)^2.$$ (42)

See Figures 2, A-C, and 3, A-C, curves e, for the predicted behavior for the rat.

**Nonhomogeneous Valanis-Landel Model**

All the LV models presented above tacitly assume that the LV wall is homogeneous, i.e., the stress-stretch relation is the same everywhere in the wall. In this model, the constant elastic modulus, C, used in the Valanis-Landel model, is replaced by an elastic modulus, $C(R)$, that is a function of radial position. The stress-stretch relation becomes:

$$S_r = C(R)A_r + H.$$ (43)

The particular $C(R)$ used here is one which makes the wall stress constant at the maximum normal diastolic pressure (12 g/cm² for the rat) and simultaneously predicts the same $V/V_0$ as found in the test data for that pressure (for the rat, $V/V_0 = 1.71$ for $P = 12$ g/cm²). (For a plot of $C(R)$ for the rat, see Figure 5.) For the predicted LV P-V and P-S relations, see Figures 2, A-C, and 3, A-C, curves f.

**Discussion**

The LV P-V and P-S relations shown in Figures 2, A-C, and 3, A-C, which are predicted by the Laplace, Lamé, Valanis-Landel and Rivlin-Saunders models, all are based on the same geometrical and material data. All of these models assume that the LV wall is isotropic and homogeneous. Therefore, any disagreement among these models in predicted LV behavior must be caused by the different approximations and assumptions inherent in each of these models and not by differences in modeling data.

The LV P-V relations, Figure 2A, predicted by the first four models, curves b through e, are markedly different from each other.

The Laplace model LV P-V relation is not correct, since the thinness condition (Relation 6), which is an integral part of that model, is not satisfied. For an LV which is bisected by an imaginary plane, force balance (neglecting acceleration effects) perpendicular to that plane requires that the LV pressure times the deformed cross-sectional area of the ventricle must equal the average wall stress perpendicular to that plane times the cross-sectional area of the wall. This force equilibrium condition must be satisfied regardless of the geometry or material properties of the LV. If the LV is a sphere and is cut diametrically by the plane, the pressure force would be $\pi r^2 P$, where $r$ is the radius of the deformed endocardium; this expression is exact regardless of the thickness of the sphere. If the sphere has a thin wall, the wall force can be approximated by $2\pi rtS$, where $t$ is the deformed wall thickness and $S$ is the average wall stress. Equating the pressure force to the approximate expression for the wall force results in the Law of Laplace for the sphere (Equation 5). Note that the Law of Laplace is an expression for the average wall stress and does not automatically imply that the wall stress is constant; it is for this reason that the wall stress predicted by the Laplace model is depicted in Figures 2, B and C, and 3, A-C, as a mark, b, on the ordinate rather than as a horizontal straight line through the thickness of the LV wall.

Since the Law of Laplace uses a pressure force term which is exact and an approximate wall force term which underestimates the cross-sectional area of the wall, it will overestimate average wall stress given LV pressure and deformed LV geometry. Conversely, the Law of Laplace will underestimate LV pressure given average wall stress and deformed LV geometry. The error of both of these estimates

**Figure 5** Variation of elastic modulus through the LV wall used in the nonhomogenous Valanis-Landel model. (Wall stress is constant at a LV pressure of 12.0 g/cm².) The horizontal dashed line represents the constant elastic modulus used in the Valanis-Landel model.
increases the greater the LV wall thickness is relative to the radius of the endocardium. One consequence of overestimating the average wall stress is that the predicted LV P-V behavior is stiffer than the actual LV P-V behavior. This conclusion is borne out by a comparison of curves a and b of Figure 2A (assuming curve a is at least approximately representative of actual behavior). The accuracy of the Laplace model is compromised further by use of the thinness condition to simplify the incompressibility condition (changing Equation 9 to Equation 10) and the wall stretch relation (Equation 11). To improve the accuracy of the Laplace model, the exact expression for the LV wall cross-sectional area, \( \pi(2rt + t^2) \), can be incorporated easily into the Law of Laplace, Equation 5, and the exact expression for the incompressibility condition, Equation 9, can be used. The third and last expression in which the thin wall approximation was used, Equation 11, where the wall stretch is equated to the wall stretch at the endocardium, cannot readily be rendered exact. As mentioned above, the Law of Laplace predicts an average wall stress; therefore, the exact expression for wall stretch must correspond to this average wall stress (see Equation 8). The determination of this stretch involves mathematical operations of a complexity comparable to those used in the Valanis-Landel model. Hence, an exact Laplace model offers no mathematical economy over exact thick-walled models and, at the same time, produces less information, since it predicts only average wall stress and not the distribution of wall stress.

The LV P-V relation of the Lamé model (Fig. 2A, curve c) is not correct, since the two assumptions inherent in that model, namely that the LV deformation is infinitesimally small and that the material properties of the myocardium can be characterized adequately by a linear engineering stress-strain relation, are not satisfied. That the Lamé model is not self-consistent can be seen from the fact that it assumes infinitesimal deformation yet predicts LV volume increases of 51% and 71% for LV pressure of 18 and 24 g/cm², respectively. Further, the endocardial strains for these pressures are 0.15 and 0.20, respectively, well beyond the range of validity of the linear engineering stress-strain relation (Fig. 4).

Many thin shell and thick shell models of the LV have been proposed (Mirskey, 1974). Most shell models, however, are approximations to spherical and nonspherical versions of the Lamé model and contain all the limitations of that model, which itself is an approximation valid only when deformations are small and the stress-strain relation is linear. Any attempt to make the Lamé model more exact by incorporating the effects of large deformation and nonlinear stress strain behavior ultimately must lead to models of the Valanis-Landel or Rivlin-Saunders type.

Although the Lamé model is grossly inadequate for finite deformation of nonlinear stress-strain material, it nevertheless is a very important model. At very small strains (low LV pressure), the assumptions upon which the Lamé model is built are satisfied; hence the P-S relation predicted by this model for low pressures must be nearly correct. Figure 2B (P-S relation for LV pressure equal to 3.0 g/cm²) shows that the Lamé model’s prediction is virtually the same as that of the more accurate Valanis-Landel and Rivlin-Saunders models. Therefore, since an LV model of complex geometry must contain a spherical geometry as a special case, and since an LV model accurate for finite deformations must also be accurate for infinitesimal deformations, these more complex models can be partially checked for error by using them to compute P-S relations for spherical LV’s at low pressure and comparing them with the Lamé prediction. If there is not good agreement for low pressures, the more complex model has some error in it.

At this juncture it should be pointed out that a popular exponential engineering stress-strain relation

\[
\sigma = a(e^{be} - 1) + H \tag{44}
\]

reduces to:

\[
\sigma = a[e^{be} -1]/e^{be/2} \tag{45}
\]

for uniaxial loading. For infinitesimal strain, Equation 45 reduces to Equation 15 where

\[
E = \frac{3ab}{2} \tag{46}
\]

Hence an LV model using the nonlinear engineering stress-strain relation 44 must predict virtually the same P-S relations for low pressures as that of the Lamé model, otherwise the nonlinear model is in error. See below for a discussion of various exponential stress-strain relations for finite deformation models.

Both the Valanis-Landel and Rivlin-Saunders models take into account the thick wall, the finite deformation, and the nonlinear stress-strain behavior of the LV. For the low LV pressure of 3.0 g/cm², both models predict virtually the same wall stress distribution (Fig. 2B, curves d and e, respectively) and, in fact, this stress distribution is virtually the same as that predicted by the Lamé model (curve c) since, at this low pressure, the assumptions of that model are nearly satisfied. Note that despite this agreement in predicted wall stress, the predicted volume changes differ significantly (Fig. 2A, curves d, e, and c, at P = 3 g/cm²). As the LV pressure is increased, however, the wall stress distributions (Figs. 2C and 3, A-C) predicted by these three models (curves d, e, and c, respectively) diverge sharply from each other, especially at the endocardium. The limitations of the Lamé model already have been discussed. The reason that the
predictions of the Valanis-Landel and Rivlin-Saunders models differ from each other is that the stress-stretch behavior of a nonlinear isotropic material cannot be characterized completely by uniaxial stress-stretch data alone.

Figure 1 shows that the uniaxial behavior of the Valanis-Landel (curve a) and Rivlin-Saunders (curve b) materials are in close agreement with each other and the test data. The biaxial behavior of these two materials, however, are radically different, and this difference is the sole cause for the difference in the predicted LV P-V relations (Fig. 2A, curves d and e, respectively) and the predicted LV P-S relations (Figs. 2C and 3, A-C, curves d and e).

The general Valanis-Landel and Rivlin-Saunders relations are each a large family of stress-stretch relations for incompressible, isotropic solids; the particular relations used in the above models are but one member of each of these families. In addition, these families of relations are just two of many (Hart-Smith, 1966; Oden, 1972). This multiplicity of possible stress-stretch relations for myocardial tissue requires multiaxial testing for full characterization of the material. For an incompressible linear material, a uniaxial test is sufficient to determine the one unknown material constant (Young's modulus); for a nonlinear material, multiaxial testing is necessary, since the functional form of the stress-stretch relation must be established in addition to its material constant(s). [Many of these nonlinear stress-stretch relations were introduced in works applied to rubbery materials; this, however, does not limit the use of these relations for biological materials in any way. Not all conceivable relations are acceptable relations according to the laws of mechanics. The above works were concerned primarily with developing mechanically acceptable stress-stretch relations; the application of these acceptable relations to rubbery materials was incidental. Whether a particular acceptable relation can represent adequately the material behavior of a given substance (biological or otherwise) can be determined only by comparing the relation's stress-stretch behavior with test data for the material.]

Direct determination of the stress-stretch relation for myocardial tissue in multiaxial states of stress has not yet been accomplished since the three dimensional curvilinear net arrangement of the fibers comprising the myocardium makes it difficult to isolate a suitably homogeneous, rectilinear portion to serve as a test specimen. An indirect determination of this relation, including anisotropic effects, can be accomplished by mathematically constructing such a relation from knowledge of the position, orientation, connectivity, and material properties of the individual fibers comprising the myocardium. For this indirect approach the work of Streeter (Streeter, 1969; Streeter and Bassett, 1966; Streeter and Hanna, 1973a, 1973b; Streeter et al., 1969, 1970) is particularly apropos; however, much work remains to be done before an accurate stress-stretch relation for myocardial tissue under multiaxial states of stress is produced.

The differences between the Valanis-Landel and Rivlin-Saunders predictions also illustrate the limitations of the common practice (Glantz, 1976; Rackley, 1976; Fester and Samet, 1974; Mirsky and Parmley, 1973; Lafferty et al., 1972; Diamond et al., 1971) of attempting to deduce myocardial stress-stretch behavior from observed LV P-V behavior. Just as two (or more) stress-stretch relations can have virtually the same uniaxial stress-stretch behavior but lead to very different P-V and P-S relations, so too can many stress-stretch relations lead to the same P-V behavior but have very dissimilar uniaxial stress-stretch behavior and lead to different P-S relations. Further, not only is the stress-stretch relation determined from P-V behavior not unique, its accuracy also is compromised by the necessity of assuming an idealized LV geometry to facilitate the computations.

A comparison of the LV P-V relation (Fig. 2A) predicted by the Valanis-Landel model (curve d) and Rivlin-Saunders model (curve e) with that taken as representative of actual behavior (curve a) suggests that the former model is the better of the two. In fact, the agreement of the Valanis-Landel model's predictions with observed LV P-V behavior is remarkable, since the model is based on a spherical geometry and assumes that the stress-stretch relation for myocardial tissue is homogeneous (same for all points of the LV wall) and isotropic (same for all directions at any point of the LV wall). This good an agreement may also be fortuitous since the data of curve a from the potassium-arrested heart may only approximate the actual behavior of the live heart in diastole.

The particular Valanis-Landel relation used here is equivalent to a popular exponential stress-natural strain relation (see, for example, Rankin et al., 1977). If, in Equation 44, stress is taken to be true stress, S, and strain is taken to be the so-called natural strain:

\[ e = \ln \lambda \]  

the Equation 44 for uniaxial strain reduces to the Valanis-Landel relation for uniaxial strain (Eq. 21) where:

\[ C = a \]  
\[ a = b. \]

Note that the Valanis-Landel and exponential stress-strain relations were compared for the same state of stress—uniaxial stress. Stress-strain relations for materials in different states of stress should not be directly compared. In general, the same material will exhibit a different stress-strain relation in a multiaxial state of stress than in the uniaxial
stress state. For example, compare the Valanis-Landel relation for the uniaxial stress state, Equation 21, with the relation for the same material in a biaxial stress state, Equation 8. Further, the same material will have different stress-strain relations in different multiaxial states of stress. For example, if the LV is modeled as a prolate ellipsoid and stress-strain relations are derived from equatorial values of circumferential stress and strain, the same material will exhibit different "material" constants for prolate ellipsoids with different major to minor axes ratios. The reason for this is that different shapes of prolate ellipsoids will result in different multiaxial states of stress even though the material and pressure loading are the same.

Another exponential stress-strain relation of the form of Equation 44 uses true stress, S (Eq. 4), for stress and engineering strain (Eq. 1) for the measure of strain (Mirskey, 1976, page 301). A third version (mentioned above in the Lame model discussion) uses engineering stress and engineering strain (Mirskey and Parmley, 1973). Neither of these last two versions is mathematically equivalent to the Valanis-Landel relation used here. However, the stress-strain data upon which they are based can always be converted to true stress-natural strain state. For example, compare the Valanis-Landel relation for the uniaxial stress state, Equation 21, with the relation for the same material in a biaxial stress state, Equation 8. Further, the same material will have different stress-strain relations in different multiaxial states of stress. For example, if the LV is modeled as a prolate ellipsoid and stress-strain relations are derived from equatorial values of circumferential stress and strain, the same material will exhibit different "material" constants for prolate ellipsoids with different major to minor axes ratios. The reason for this is that different shapes of prolate ellipsoids will result in different multiaxial states of stress even though the material and pressure loading are the same.

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The last model presented here extends the most successful of the previous four, the Valanis-Landel model, to study the effects nonhomogeneity might have on predicted LV behavior. So far, the only material data used are those from uniaxial tests of papillary muscle, which do not shed any light on possible nonhomogeneity of the myocardium. Although the LV P-V relation predicted by the Valanis-Landel model agrees well with observed behavior, the predicted wall stress distributions, Figures 2, B and C, and 3, A-C, curves d, do not appear to be physiologically reasonable, since only tissue near the endocardium is highly stressed, whereas the majority of tissue in the wall is hardly stressed at all even at a diastolic pressure of 24 g/cm² (Fig. 3C) which is twice the normal maximum diastolic pressure for the rat. It seems unreasonable that myocardial tissue should be used so inefficiently. Optimal efficiency suggests that the wall stress should be constant through the thickness of the wall, and examination of the Valanis-Landel model shows that this can be accomplished for a single LV pressure by varying the elastic modulus with radial position. The particular variation of elastic modulus used in the nonhomogeneous Valanis-Landel model was determined by requiring the wall stress to be constant at 12 g/cm², the maximum normal diastolic pressure for the rat, and also that the predicted and observed LV P-V relations coincide at that pressure. Figure 5 shows that the elastic modulus monotonically increases from the endocardium to the epicardium and that the constant elastic modulus used in the Valanis-Landel model, which was independently obtained from uniaxial papillary muscle tests, lies within the range of the predicted nonhomogeneous elastic modulus. Further, the predicted lower elastic modulus at the endocardium is consistent with the presence of muscular ridges on an actual endocardium, since by idealizing the irregular surface of an actual endocardium as a spherical surface, as is done in the model, the effective elastic modulus near the endocardium of the model would be less than the elastic modulus of the tissue comprising the muscular ridges.

Figures 2, B and C, and 3, A-C, curves f, show the predicted wall stress distribution for various values of LV pressure. Note that, for pressures below the maximum normal diastolic pressure, the predicted wall stress increases from endocardium to epicardium. This is qualitatively different from the predictions of the homogeneous model (curve d). For pressures above the maximum normal diastolic pressure, the predicted wall stress decreases from endocardium to epicardium; this is qualitatively similar to the predictions of the homogeneous model, but the gradation of wall stress is smoother in the nonhomogeneous model than in the homogeneous one; the predicted LV behavior appears to use the myocardial tissue more effectively at all pressure levels.

Note that this model can be thought of as an attempt to account for the irregular endocardial geometry while preserving the simplicity of a spherical model. The nonhomogeneous elastic modulus need not be taken literally.

In summary, the model of LV mechanics based on the Law of Laplace is shown to use the thin wall approximation in three distinct ways: in calculating the cross-sectional area of the LV wall, in imposing the incompressibility condition for myocardial tissue, and in deriving an expression for effective wall stretch. This thin wall approximation leads to an overestimation of wall stress and wall stiffness. Further, it is demonstrated that the assumptions of small deformation and a linear stress-strain relation for myocardial tissue, which are used in the Lame model, are not consistent with LV mechanics. The findings from a detailed examination of these two models suggest that the minimum prerequisites for a mathematical model of LV mechanics are that it include the effects of the thick wall, large defor-
mation, and nonlinear stress-stretch behavior of the LV. Next, a Valanis-Landel model and a Rivlin-Saunders model are presented; each of these has the above minimum prerequisites. A comparison of LV behavior predicted by these two models graphically demonstrates that uniaxial stress-stretch data of papillary muscle are insufficient to characterize the multiaxial stress-stretch behavior of myocardial tissue. On the basis of a comparison of predicted with observed LV P-V behavior, the Valanis-Landel model appears to be the better of the two. Finally, the Valanis-Landel model is modified to explore what degree of nonhomogeneity of the myocardium is necessary for the wall stress to be constant at the maximum normal diastolic pressure. The necessary variation in elastic modulus was found to increase monotonically from the endocardium to the epicardium and to include the value of the constant elastic modulus (obtained independently from uniaxial tests of papillary muscle) which was used in the homogeneous Valanis-Landel model.

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References


Hart/Smith LJ (1966) Elasticity parameters for finite deformations of rubber-like materials. Table I. ZAMP 17: 608-625


Janz RF, Grimm AF (1972) Finite element model for the mechanical behavior of the left ventricle. Circ Res 30: 244-247


Rackley CB (1976) Quantitative evaluation of left ventricular function by radiographic techniques. Circulation 54: 862-879


Spotnitz HM, Sonnenblick EH (1973) Structural conditions in the hypertrophied and failing heart. Am J Cardiol 32: 398-406


Woods RH (1892) A few applications of a physical theorem to membranes in the human body in a state of tension. J Anat Physiol 26: 362-370
The law of Laplace. Its limitations as a relation for diastolic pressure, volume, or wall stress of the left ventricle.

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