Finite-Element Model for the Mechanical Behavior of the Left Ventricle

PREDICTION OF DEFORMATION IN THE POTASSIUM-ARRESTED RAT HEART

By Ronald F. Janz and Arthur F. Grimm

ABSTRACT
A finite-element model is used to analyze the mechanical behavior of the left ventricle. The ventricle is treated as a heterogeneous, linearly elastic, thick-walled solid of revolution. The inner third of the ventricular wall is assumed to be transversely isotropic with a longitudinal Young's modulus, transverse modulus, and shear modulus of 60 g/cm², 30 g/cm², and 15.5 g/cm², respectively. In the outer two-thirds of the ventricular wall the myocardium is assumed to be isotropic with a Young's modulus of 60 g/cm². Poisson's ratio is assumed to be equal to 0.45 throughout the ventricular wall. The valvular ring at the base of the ventricle is simulated by a homogeneous layer of collagen. The model appears to predict gross free-wall deformation in the left ventricle of the potassium-arrested rat heart fixed in situ. The presence of a relatively compliant transversely isotropic region near the endocardial surface results in significantly lower axial and circumferential stresses in this region than are present in a homogeneous, isotropic model. The presence of a simulated valvular ring results in a concentration of relatively large stresses near the base of the ventricle.

KEY WORDS axial symmetry of myocardial stress and strain endocardial fiber elongation linearly elastic myocardium simulated valvular ring transversely isotropic trabeculae passive papillary muscle elasticity

Several models which have been proposed for the mechanical behavior of the left ventricle were reviewed briefly in a recent paper by Falsetti et al. (1). Even though these models may yield relatively simple mathematical expressions for stress (tension/unit area) in terms of pressure and dimensional parameters, their predictive capabilities are limited by simplifying assumptions with regard to ventricular geometry and composition.

In this study a model for the mechanical behavior of the left ventricle is proposed which is based on less restrictive assumptions than those inherent in previous models. The geometry of the proposed model is similar to a truncated, thick-walled ellipsoid of revolution. The plane of truncation in the model corresponds to the base of the ventricle. The wall of the model near this plane consists of an isotropic material with elastic properties comparable to those of collagen. The remainder of the model corresponds to the ventricular wall extending from the base to the apex. This part of the model is divided into two homogeneous, linearly elastic layers. The inner layer consists of a transversely isotropic material which simulates the combination of muscle and sinusoidallike passages near the endocardium of the ventricle. The outer layer consists of an isotropic material which simulates the...
MODEL FOR THE LEFT VENTRICLE

region of densely packed muscle fibers near the epicardium of the ventricle.

The radial, axial, circumferential, and shear components of stress and strain (fractional change in length) relevant to the axisymmetric deformation of a thick-walled solid of revolution are determined by the model. In addition, stress and strain in a variable direction approximating the local endocardial fiber direction in the left ventricle are also determined. Because of the detail which is taken into account by the model, a numerical approach involving a digital computer is used. The numerical approach selected is based on the finite-element method (2).

A preliminary assessment of the predictive capability of the model is made by comparing the wall geometry of the model with the free-wall geometry of the potassium-arrested left ventricle of the rat at corresponding pressures. Implications of a separate region in the model for the valvular ring and a heterogeneous, transversely isotropic model for the myocardium are then discussed.

Methods

Serial sections of hearts of adult Sprague-Dawley albino male rats were used to determine the free-wall geometry of the left ventricle. The rats were anesthetized with sodium pentobarbital, 50 mg/kg body weight. The serial sections were obtained from open-chest animals at controlled transmural pressures after potassium arrest and formalin fixation in situ. A typical section is shown in Figure 1. The plane of this section is perpendicular to the apex-valve axis of the left ventricle. This axis was established by positioning a stainless steel wire between the leaflets of the aortic valve and the apical dimple. The center of the left ventricle in the plane of a given serial section was taken to be the point at which all lumen radii were approximately equal. The exterior radius, r, of the free wall of the left ventricle in this plane was measured with respect to the center point. The corresponding interior

![Figure 1](http://circres.ahajournals.org/)

A serial section of a potassium-arrested rat heart fixed in situ illustrating the geometric symmetry of the left ventricle.
radius, \( r_j \), was determined by equating the observed cross-sectional area of the lumen to \( \pi r_j^2 \). Two radii were obtained from each serial section. The collection of all radii pairs from all serial sections was used to define the free-wall geometry at a given intraventricular pressure. The preparation of the serial sections and the determination of the left ventricular free-wall geometry from these sections are discussed in more detail by Klein (3).

The proposed model assumes that the left ventricle is a solid of revolution. This assumption is based on the observed geometry of the left ventricle in the potassium-arrested rat heart (Fig. 1). To simplify the computational aspects of the study the assumption is also made that the left venticle deforms axisymmetrically. Unfortunately, this assumption precludes an accurate representation of the fibrous structure of the myocardium. However, it is possible to include some degree of anisotropy within the limitations of axisymmetric deformation.

Streeter et al. (4, 5) have made several pertinent observations with regard to fiber orientation. (1) In the systolic left ventricle of the dog, fiber direction in the intraventricular septum, the interior cardiac wall, the left cardiac wall, and the posterior cardiac wall exhibits roughly the same dependence on distance (normalized with respect to wall thickness) from the epicardial surface. This observation is supported by measurements of fiber orientation in the vicinity of the left cardiac wall of the diastolic and systolic left ventricle of the dog. (2) Significant changes in fiber angle between diastole and systole do not occur. (3) The angle between the dominant fiber direction and circumferential direction appears to vary continuously from approximately \(-90^\circ\) at the epicardial surface to \(+90^\circ\) at the endocardial surface. (4) The majority of the muscle fibers appear to lie in concentric surfaces parallel to the epicardial surface.

The first two of these observations can be accounted for in the proposed model by assuming that fiber direction is independent of the circumferential coordinate and the state of deformation. The third observation, however, appears to be inconsistent with axisymmetric deformation and is not taken into account by the model. The fourth observation is accounted for in a limited way by assuming that the myocardium is transversely isotropic in the inner third of the ventricular wall. The principal axes are assumed to be coincident with coordinate directions established by tangent and normal vectors at the epicardial surface. Young's modulus along the tangent to the epicardial surface is assumed to be equal to the small-strain modulus of passive left ventricular papillary muscle of the rat under uniaxial load. The transverse modulus is assumed equal to one-half of the passive papillary muscle modulus. The latter somewhat arbitrary choice of modulus is based on the observation that sinusoidal-like spaces compose a significant proportion of the wall in the vicinity of the endocardium. As the intraventricular pressure is increased, it is apparent from the serial sections of rat hearts discussed earlier that the muscle fibers become more densely packed with a pronounced decrease in thickness of this region.

The myocardium is assumed to be isotropic in the outer two-thirds of the ventricular wall. This assumption is based on the observation that muscle fibers in this region appear to be densely packed relative to the inner third of the ventricular wall. Assuming that individual muscle fibers are isotropic, the outer two-thirds of the wall would tend to be isotropic at least with respect to gross deformation.

For a linearly elastic, transversely isotropic material which deforms axisymmetrically, stress is related to strain by a system of linear equations of the form:

\[
\begin{bmatrix}
\sigma_{\tau p}
\end{bmatrix}_{\tau p} = [C] \begin{bmatrix}
\epsilon_{\tau p}
\end{bmatrix}_{\tau p},
\tag{1}
\]

where \( \{\sigma\} \) and \( \{\epsilon\} \) denote stress and strain vectors, respectively, referred to the principal axes of the material (denoted here by \( t, p, \) and \( \theta \)) and \([C]\) is a \( 4 \times 4 \) symmetrical matrix of the form:

\[
\begin{align*}
C & \equiv \begin{bmatrix}
\epsilon^0 - 1
\end{bmatrix},
\end{align*}
\]

where \( \epsilon \equiv \text{stress}; \, \epsilon = \text{strain}; \, \sigma \, \text{and} \, \beta = \text{constants.}
\]

For \( \epsilon << 1 \), \( \epsilon (\epsilon^0 - 1) \approx \sigma \beta + 0 (\epsilon^2) \). The product \( \sigma \beta \) denotes the small-strain modulus.

The relation between elastic stress and strain for passive left ventricular papillary muscle of the rat under uniaxial load can generally be represented by an exponential expression of the form:

\[
\sigma \equiv a (\epsilon^0 - 1),
\]

where \( \sigma \equiv \text{stress}; \, \epsilon \equiv \text{strain}; \, a \) and \( \beta \) = constants.

For \( \epsilon << 1 \), \( a (\epsilon^0 - 1) \approx \sigma \beta + 0 (\epsilon^2) \). The product \( \sigma \beta \) denotes the small-strain modulus.

\[
\begin{align*}
E_t \, (\text{g/cm}^2) & \quad 60 \quad 60 \\
E_p \, (\text{g/cm}^2) & \quad 30 \quad 60 \\
\rho_p & \quad 0.45 \quad 0.45 \\
\tau_p & \quad 0.45 \quad 0.45 \\
G_{tp} \, (\text{g/cm}^2) & \quad 13.5 \quad 20.7
\end{align*}
\]

\[E = \text{Young's modulus}; \, G = \text{shear modulus}; \, \rho = \text{Poisson's ratio}; \, t, p, \theta = \text{axes of the surface coordinate system.}\]
and $E$, $G$, and $\nu$ denote Young’s modulus, shear modulus, and Poisson’s ratio, respectively. Assuming the myocardium to be isotropic is equivalent mathematically to assuming the conditions:

$$ E_t = E_p, 
\nu_t = \nu_{ph}, 
C_{tp} = \frac{E_t}{2(1 + \nu_{ph})}. $$

The values of the elastic constants chosen for this study are shown in Table 1. The small-strain modulus under uniaxial load ($E_t$) is taken to be equal to 60 g/cm². The shear modulus ($15.5$ g/cm²) for the transversely isotropic material was obtained by dividing the average of the two Young’s moduli ($45$ g/cm²) by the quantity $2(1 + \nu_{ph}) = 2.9$. Poisson’s ratio for the myocardium is assumed to be equal to 0.45. For the range in interior hydrostatic pressures considered in this study this corresponds to less than 10% change in volume under load except in the immediate vicinity of the apex and base of the ventricle.

By using the laws of transformation for the stress and strain tensors, it can be shown that the relations between stress and strain in terms of the datum (cylindrical) coordinates are of the form:

$$ \{\sigma\}_{r\theta} = [t]^T [C] [t] \{\epsilon\}_{r\theta} \quad (2) $$

where $\{\sigma\}_{r\theta}$ and $\{\epsilon\}_{r\theta}$ denote vectors whose elements consist of the radial, axial, circumferential, and shear components of stress and strain, respectively. The superscript, $T$, denotes the transpose of the matrix, $[t]$. A derivation of the $4 \times 4$ transformation matrix, $[t]$, used in this study is given in the appendix.

The state of deformation of an elastic solid subjected to an external load can be predicted by solving a boundary-value problem consisting of the equilibrium equations formulated in terms of displacement together with the prescribed boundary conditions. Alternately, the state of deformation of the solid can be predicted by solving an equivalent variational problem. The latter approach is suggested by the minimum potential energy theorem (small-displacement theory which states that those displacements which satisfy the prescribed boundary conditions and minimize the potential energy of the solid are, in fact, the equilibrium displacements of the solid (6). The finite-element method is a special case of the Rayleigh-Ritz procedure for constructing a minimizing sequence for a functional (7). The functional in this case is potential energy and the minimizing sequence is a sequence of functions which represents estimates of displacements in the solid.

In the finite-element method, the minimizing sequence is constructed in the following way. The elastic solid is first subdivided into a finite
A typical finite element of revolution in the subdivision shown in Figure 2. The nonzero components of stress and strain relevant to axisymmetric deformation are superimposed on the element. See text for abbreviations and discussion.

number of discrete elements. A subdivided model for the left ventricle is shown in Figure 2. A typical finite element in this subdivision is shown in Figure 3. The radial (r) and axial (z) components of displacement within each element are assumed to be first degree polynomials in the r and z coordinates of the datum coordinate system. The coefficients of these polynomials are uniquely related to the displacements of the circumferential joints or nodal circles of each element. The potential energy (V) of the solid in terms of these nodal displacements is of the form:

\[
V = \frac{1}{2} \{u\} ^T [K] \{u\} - \{u\} ^T \{Q\},
\]

(3)

where \(\{u\}\) denotes the displacement vector consisting of the r-z components of displacement of each nodal circle in the subdivided solid, \([K]\) denotes the positive definite-stiffness matrix, and \(\{Q\}\) is the load vector determined by external loads on the solid. A necessary condition for Eq. 3 to have a minimum is that:

\[
\frac{\partial V}{\partial u_i} = 0 \quad i = 1, 2, \ldots, N,
\]

(4)

where \(N\) denotes the number of components in the vector, \(\{u\}\). A system of linear equations for \(u_i, i = 1, 2, \ldots, N\) is obtained by differentiating Eq. 3 according to Eq. 4; namely,

\[
[K] \{u\} = \{Q\}.
\]

(5)

Better estimates of \(\{u\}\) can normally be obtained by subdividing the solid into smaller finite elements. The accuracy of the assumed linear displacement field within each element normally increases with decreasing element size. The minimum value of Eq. 3, in turn, decreases as the estimates of \(\{u\}\) improve.

A planar view of the proposed finite-element model for the left ventricle is shown in Figure 4. As indicated in this figure, the free-wall geometry of the potassium-arrested left ventricle of the rat at an intraventricular pressure equal to 0 cm H\textsubscript{2}O defines the geometry of the model. There are 198 finite elements in the model, each of which has a quadrilateral cross section. This subdivision was maintained under all loading conditions discussed. A relatively large number of elements are chosen to reduce numerical errors associated with the representation of the continuum as a series of discrete elements and to obtain the desired spatial resolution in stress and strain. The computer program which is used to compute stress and strain in each element of this model was developed at this laboratory (8). Numerical results obtained with this program are presented in the next section.
Results and Discussion

In Figures 5-7, the predicted shapes of potassium-arrested left ventricles of the rat at intraventricular pressures of 3, 6, and 12 cm H$_2$O are compared with shapes reconstructed from serial sections of rat hearts. As indicated in these figures, the wall geometry of the model agrees well with the free-wall geometry of the left ventricle of the rat. These comparisons, of course, are only a preliminary step in the validation of the model. A more accurate assessment of the predictive capability of the model could be made by making similar comparisons between observed and computed values of stress and strain within the left ventricular wall of the rat heart. However, the appropriate data for making these comparisons do not appear to be available.

One of the limitations of the model is the assumption that the myocardium is linearly elastic. This limitation is apparent in Figure 8 which shows that the model overestimates deformation of the ventricular wall at an intraventricular pressure of 24 cm H$_2$O. This deviation between predicted and observed behavior is probably due to the nonlinear elastic response of the myocardium. The passive stiffness of left ventricular papillary muscles...
A comparison of predicted and observed free-wall geometry of the potassium-arrested left ventricle of the rat at an intraventricular pressure of 24 cm H₂O.

Muscle increases with increasing muscle length. A linear model for the myocardium based on an elastic modulus for papillary muscle which is applicable under small preloads would therefore overestimate deformation at large preloads. In addition, the effective transverse modulus near the endocardial surface probably increases as muscle becomes more densely packed in this region with increasing intraventricular pressure.

The inner third of the wall of the model proposed in this study is transversely isotropic with respect to a direction which corresponds, roughly, to the fiber direction near the endocardial surface. Strain in this direction should, therefore, be a measure of the incremental change in endocardial fiber length. For a given intraventricular pressure the model predicts an endocardial fiber elongation at the equator which is a factor of ten less than that at the apex and a factor of twenty less than that at the base.

The relatively large amount of deformation which occurs near the simulated valvular ring in the model is also reflected in the myocardial stress distributions. The model predicts a concentration of relatively large stresses in the immediate vicinity of the base of the ventricle. Stresses within the valvular region itself are also generally higher than those predicted for the myocardium.

The deformed state of the model is significantly affected by the degree of heterogeneity and anisotropy which is used to represent the myocardium. If it is assumed that the entire ventricular wall consists of a single homogeneous, isotropic material (E = 60 g/cm², ν = 0.45), the overall predicted deformation for a given intraventricular pressure is less than that indicated in Figures 5-8. For example, the lumen radius at the equator of the homogeneous, isotropic model with an internal hydrostatic pressure of 12 cm H₂O is approximately 8% less than the corresponding lumen radius in the proposed model. As expected, the differences in terms of stress or strain are much more significant. For example, near the intersection of the equatorial plane...
and endocardial surface the homogeneous, isotropic model predicts values for circumferential and axial stress which exceed corresponding values predicted by the proposed model by factors of two and three, respectively.

A more favorable comparison with the state of deformation in the proposed model is obtained if the inner third of the ventricular wall is assumed to be isotropic, but more compliant \( E = 30 \, \text{g/cm}^2, \, v = 0.5 \) than the outer two-thirds of the wall. In this case the lumen radii at the equator, with an internal hydrostatic pressure of 12 cm H\(_2\)O, differ by less than 1%. In fact, the dimensions of the entire ventricular wall predicted by either model agree within 1%. However, the heterogeneous isotropic model predicts significantly lower values for circumferential and axial stresses in the inner third of the ventricular wall than does the proposed model.

### Appendix

**DERIVATION OF TRANSFORMATION MATRIX FOR STRESS-STRAIN RELATIONS IN A TRANSVERSELY ISOTROPIC MATERIAL**

**Datum Coordinate System.**—The datum coordinate system implied by the assumption of axisymmetric geometry and deformation is the cylindrical coordinate system shown in Figure 9. The \( z \)-axis coincides with the apex-valve axis. The radial and circumferential coordinates are denoted by \( r \) and \( \theta \), respectively.

**Surface Coordinate System.**—A second coordinate system is introduced for convenience at this point. This system is denoted by the ordered triple \( (t, \theta, p) \) also illustrated in Figure 9. The \( t \)- and \( p \)-axes lie in a plane which contains the apex-valve axis. The origin of this coordinate system is located on the arc determined by the intersection of this plane with the epicardial surface. The \( t \)- and \( p \)-axes are coincident with lines which are tangent and perpendicular to this arc at a point which is located at distance, \( l \), along the arc from the ventricular apex.

**Transformation of the Stress Tensor from Surface Coordinate System to Datum Coordinate System.**—The nonzero components of the stress (and strain) tensor in axisymmetric deformation are shown in Figure 3. As shown in this figure, there are four such components, namely, radial stress \( \sigma_{rr} \), axial stress \( \sigma_{\theta\theta} \), circumferential stress \( \sigma_{\theta z} \) and shear stress \( \tau_{rz} \).

The components of the stress tensor in two different orthogonal coordinate systems are related in the following way (6):

\[
\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} (\vec{e}_k \cdot e_l) \sigma_{kl} \tag{A1}
\]

where, \( \sigma_{ij} \) and \( \overline{\sigma}_{kl} \) denote stress tensors in the two coordinate systems and \( e_i, \, i = 1, \, 2, \, 3 \) and \( \vec{e}_k, \, k = 1, \, 2, \, 3 \) denote unit vectors coincident with the three coordinate directions in each of the coordinate systems. The notation \((\cdot)\) is used to denote the dot product of two vectors. In this discussion \( e_1 = e_t, \, e_2 = e_p, \, e_3 = e_\theta, \, e_1 = e_t, \, e_2 = e_p, \, e_3 = e_\theta, \, e_3 = e_\theta, \, \sigma_{11} = \sigma_{tt}, \, \sigma_{22} = \sigma_{pp}, \, \sigma_{33} = \sigma_{\theta\theta}, \, \sigma_{12} = \tau_{tp}, \, \sigma_{13} = \sigma_{tr}, \, \sigma_{23} = \sigma_{pz}, \, \sigma_{33} = \sigma_{\theta z}, \, \sigma_{12} = \tau_{rz}. \)

Since the model deforms axisymmetrically under internal hydrostatic pressure, \( \tau_{tp} = \tau_{tr} = \tau_{pz} = 0. \)

The coefficients of \( \sigma_{kl} \) in Eq. A1 are determined by relating the unit vectors from each coordinate system to each other. This is easily done in the following way. The position of the origin of the surface \( (t, \theta, p) \) coordinate system in terms of cylindrical coordinates is \( (r(l), \, z(l)) \).

As indicated by this notation the coordinates of a point on the epicardial surface depend parametrically on arc length, \( l \). It then follows that

\[
e_t = r'(l)e_r + z'(l)e_z, \quad e_p = -z'(l)e_r + r'(l)e_z. \tag{A2}
\]

The primes denote differentiation with respect to arc length.

The desired transformation matrix is obtained by substituting Eq. A2 into Eq. A1. Using the symmetry of the stress tensor, the result of this substitution is:

\[
\left\{ \sigma \right\}_{r\theta} = \left[ t \right]^T \left\{ \sigma \right\}_{tp}, \tag{A3}
\]

where

\[
\left[ t \right]^T = \begin{bmatrix}
\sigma_{rr} & \sigma_{rt} \\
\sigma_{tr} & \sigma_{tt}
\end{bmatrix}, \quad \left\{ \sigma \right\}_{tp} = \begin{bmatrix}
\sigma_{rt} \\
\sigma_{tp}
\end{bmatrix}
\]

and

\[
\left[ r'(l) \right]^2 \begin{bmatrix}
\sigma_{rr} & \sigma_{rt} \\
\sigma_{tr} & \sigma_{tt}
\end{bmatrix} \begin{bmatrix}
r'(l) z'(l) \\
0
\end{bmatrix} = \\
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
r'(l) \sigma_{rt} \\
-\sigma_{rr} \, z'(l) \, l + 2 \sigma_{rt} \, l^2
\end{bmatrix}.
\]

The values of \( r'(l) \) and \( z'(l) \) at the centroid of each finite element in the inner third of the ventricular wall are determined by numerical differentiation of the indicated parametric representation of the epicardial surface at an intraventricular...
pressure of 0 cm H₂O. The appropriate value of \( l \) (arc length along the epicardial surface) for each element in this region is determined by numerically constructing a line which passes through the centroid and intersects the epicardial surface perpendicularly. This can be done in a unique way for all elements except those which are near the valvular ring. For these elements the smallest value of arc length which satisfies the above criterion is chosen.

Acknowledgment
The authors acknowledge the assistance of A. H. Gott in preparing the computer graphics and S. A. Klein, K. V. Katele, H. L. Lin, and T. Hebda for preparing the serial sections of the rat hearts and determining the free-wall geometry of the left ventricle from these sections.

References
Finite-Element Model for the Mechanical Behavior of the Left Ventricle: PREDICTION OF DEFORMATION IN THE POTASSIUM-ARRESTED RAT HEART

RONALD F. JANZ and ARTHUR F. GRIMM

_Circ Res._ 1972;30:244-252
doi: 10.1161/01.RES.30.2.244

_Circulation Research_ is published by the American Heart Association, 7272 Greenville Avenue, Dallas, TX 75231
Copyright © 1972 American Heart Association, Inc. All rights reserved.
Print ISSN: 0009-7330. Online ISSN: 1524-4571

The online version of this article, along with updated information and services, is located on the World Wide Web at:
http://circres.ahajournals.org/content/30/2/244

Permissions: Requests for permissions to reproduce figures, tables, or portions of articles originally published in _Circulation Research_ can be obtained via RightsLink, a service of the Copyright Clearance Center, not the Editorial Office. Once the online version of the published article for which permission is being requested is located, click Request Permissions in the middle column of the Web page under Services. Further information about this process is available in the Permissions and Rights Question and Answer document.

Reprints: Information about reprints can be found online at:
http://www.lww.com/reprints

Subscriptions: Information about subscribing to _Circulation Research_ is online at:
http://circres.ahajournals.org/subscriptions/