The problems of characterizing pulsatile patterns of pressure and flow in the arterial system are intriguing and complex. Ejection of blood from the left ventricle initiates non-linear transients in pressure and flow at the root of the aorta. These transients initiate complex pulse patterns that are propagated throughout the arterial tree. Some of the factors that must be reckoned with in an analysis of these patterns are itemized in the list that follows: 1) The force initiating the transients is itself complex; the velocity of ventricular ejection increases rapidly with the opening of the aortic valve, then declines slowly to reach a negative nadir with the closure of the aortic valve. 2) The distensibility of the walls of the arteries receiving this positive increment of pressure and flow has an important influence on pulse patterns. This physical property of the arterial wall is also responsible for changes in configuration and velocity of the transients as they pass over the arterial tree. The pulsatile patterns are further distorted by 3) frictional losses of energy with both positive and negative flow of the blood, and by the 4) branching and tapering architecture of the arterial tree. 5) The resistances to forward motion of blood through the distal arteriolar beds also have their effects on the contours of the arterial pulses observed upstream.

Both experimental and theoretical methods have been used to analyze and to quantify the influence of these factors on the arterial pulse pattern. The experimental approach has profited substantially from new instrumentation which permits a recording of the transients of both pressure and flow with a high degree of fidelity at various levels of the arterial tree. The theoretical approach employs established mathematical relations between known physical parameters which permit the prediction of pressure and flow at specific points in the arterial tree and specific times in the cardiac cycle. If an adequate mathematical expression were available to describe these relations, the fit of these predictions to the actual measured values would then become a valuable tool both for assessing the accuracy of the selected values for the physical parameters and for evaluating the significance of the various factors that influence these transients in the arterial tree.

The theoretical analysis of these transients then has two requirements: 1) realistic values for factors that influence the transients, and 2) a mathematical statement of the interrelationship of these factors which will define pressure and flow with respect to time and position in the arterial tree. Approximations of the required quantitative values for the physical factors involved can be found in the literature. However, one of the more unyielding problems has been the development of a mathematical expression for the interrelation of these factors in pulsatile flow in a distensible vessel. Owing to the difficulties encountered, greatly simplifying assumptions have been made, such as laminar flow, linear frictional resistance, and steady oscillatory flow. The resulting equations, despite the restrictive assumptions, are very complicated.
and do not lend themselves to ease of computation when practical boundary conditions are introduced.

The current study presents a mathematical approach which permits a more realistic solution of specific equations describing these interrelations. Specific values for pressure and flow with respect to time and position in the arterial tree can be computed. This approach starts with two established equations dealing with pressure and flow transients and the physical factors that influence them. These simultaneous equations are: 1) the continuity equation, which equates the net influx of blood entering a small segment of the arterial tree with the increase of volume of that segment (fig. 2), and 2) the momentum equation (Newton's second law), which equates the forward force acting on this segment of blood with the backward force plus the force exerted by the arterial wall and the force required to overcome friction in the artery (fig. 3). The inclusion in this equation of the statement for friction makes it a nonlinear, partial differential equation which has not been solved satisfactorily by previous methods. The critical aspect of the current approach is the application of the method of characteristics which permits the solution of these two simultaneous equations. Specific values can be computed for the unknown functions of pressure and velocity along characteristic lines relating the independent variables of time and distance (figs. 4 and 5). With the aid of a high speed computer these unknowns have been determined for frequent periods of time (1/400th of a physiological pulse cycle) and for short segments along the arterial tree (1 cm).

In developing this approach we were not aware that in 1958 Lambert applied similar principles to this problem.* Although he employed the method of characteristics, the use of a graphic solution instead of a computer, and the inadequate consideration of physical conditions in the circulatory system limited the usefulness of his attempt and led to unrealistic results.

The problems of reflections and their interpretation in an analysis of unsteady flow become less significant with this approach. A pressure pulse is transmitted through the vessel at a speed that depends upon the tube properties and the pressure within the tube. As the pressure varies both with distance along the tube and with time, the speed of the pulse wave changes continuously with the independent variables, distance along the tube and time. For any change in speed of the pulse wave, reflections are set up which move in the reverse direction, and the transmitted wave is affected as well. As reflections also occur at boundaries, i.e., entrances, exits, branches and obstructions, previous methods of keeping track of all these reflections become hopelessly complicated.

The method outlined here takes all these reflections into account automatically, not by keeping track of them separately, but by satisfying all of the basic equations at closely spaced sections along the vessel at frequent time intervals, and by satisfying the boundary equations. Owing to the relative ease of handling the computations for interior sections and for boundaries, solutions may be programmed to simulate conditions in branching arteries with satisfactory accuracy.

The theory of characteristics, which applies to the solution of hyperbolic partial differential equations, first gained prominence in solution of supersonic flow problems by Courant and Friedrichs. These methods were extended to applications of free-surface flow cases later, by Stoker. More recently they have been applied to waterhammer situations by Lai, Streeter and Lai, and Streeter, in which nonlinear terms for wall expansion and for wall friction have been retained in the equations. This paper presents an extension of the theory to flexible vessels, tapering vessels, and vessels with distributed outflow along their length, together with in vivo experimental confirmation of the method.

---

*We are indebted to the Editorial Board of Circulation Research for calling this publication to our attention.
Theoretical Methods and Results

In this section the mathematical and physical relationships for flow through flexible tubes are derived, including equations for tapering vessels with distributed outflow.

A. DERIVATION OF BASIC EQUATIONS

The assumptions required for analytical handling of pulsatile flow are first discussed, followed by development of elasticity relations and the continuity and momentum equations for vessels of constant initial diameter. From these relationships the characteristic equations are obtained and finite difference methods applied to develop the equations for the method of specified time intervals. After discussion of boundary conditions, an example is presented.

1. Basic Assumptions

The basic assumptions required in developing the working equations are:

a. One-dimensional flow; the velocity at a cross section is given by the average velocity at the section at a given instant.

b. The vessel walls are elastic, with a Poisson ratio of 0.5, and they are tethered; hence the volume of elastic vessel wall per unit length remains constant.

c. The fluid density is constant. Compressibility of blood is small compared with expansion of the vessel walls under increased pressure, and

d. Pressure losses due to wall friction may be expressed as proportional to some power of the velocity at a cross section.

2. Elasticity Relationships

Although arteries have viscoelastic properties, their effect seems to be minor and previous investigators have concluded that the stress-strain curve may be considered linear without introducing appreciable error.

If $D$ is the inside diameter of the vessel, and $h$ the effective wall thickness, then as a consequence of the constant volume of elastic wall material resulting from the assumption of Poisson’s ratio of 0.50,

$$Dh = Dh_0$$  \hspace{1cm} (1)

By using the method of analysis for stresses within the walls of thin-walled vessels, at pressure $P$, figure 1, the tensile force per unit length of tube is given by

$$2T = PD$$

since the force component on the curved half section of the tube is equal to the pressure times the projected area of the curved surface (Ref. 14, p. 48, 52). Now, considering an increment of pressure $dP$, the equilibrium equation, from figure 1, is

$$2(T + dT) = PD + d(PD)$$

as the projected area also increases by $dD$.

By taking the difference of the last two equations, and after solving for $dT$,

$$dT = \frac{d(PD)}{2}$$

Since the diameter change, on a percentage basis, is much less than the pressure change, the term $PdD/2$ is neglected in expanding the right-hand side, leaving

$$dT = \frac{D}{2} dP$$  \hspace{1cm} (2)

By dividing through by the wall thickness the change in tensile stress $dS$ (force per unit area) is obtained

$$dS = \frac{dT}{h} = \frac{DdP}{2h}$$

Now, by dividing through by the elastic modulus of the vessel wall, $Y$, the unit strain is obtained, which is the change in length per unit length caused by $dP$. Since circumference changes are proportional to diameter changes, the unit strain is $dD/D$,

$$\frac{dD}{D} = \frac{D}{2h} \frac{dP}{Y}$$  \hspace{1cm} (3)
After use of equation 1 to eliminate $h$, and after separating variables

$$dP = 2Yh_o D_o \frac{dD}{D^2}$$  \hspace{1cm} (4)

Integrating

$$P = Yh_o D_o \left( \frac{1}{D^2_o} - \frac{1}{D^2} \right)$$  \hspace{1cm} (5)

in which the condition has been used that $D = D_o$ when $P = 0$. By multiplying and dividing the right-hand side by $\pi/4$ to introduce the vessel cross sectional area $A$, and correspondingly $A_o$,

$$A/A_o = \frac{1}{1 - PD_o h_o Y}$$  \hspace{1cm} (6)

This form of equation permits the vessel to blow up when $PD_o = h_o Y$. For the normal pulse pressure range where linear elastic relationships are assumed, the pressure is below this limiting value. Peterson, Jensen and Parnell\textsuperscript{11} have demonstrated that for the physiological pressure pulse the stress: strain relationship in the wall (Young's modulus) can be considered to be constant. Burton\textsuperscript{12} and Bergel\textsuperscript{13} have, however, emphasized the physiological importance of a pressure dependence of Young's modulus in the higher pressure range. Here adventitial connective tissue adds support to the wall structure. In considering larger pressure transients such as occur in exercise it may be appropriate to use a nonlinear elastic relationship, as Lambert\textsuperscript{5} has done.

The pressure $P$ has been expressed as force per unit area ($\text{dynes/cm}^2$). It is customary to express it in terms of the height of a liquid column (the fluid flowing). These are related by $P = \rho g H$ in which $\rho$ is the mass density ($g/ml$), $g$ is gravity (980 cm/sec$^2$), $H$ is the pressure in height of fluid flowing (cm). Two additional substitutions are made, let

$$a^* = \frac{h_o Y}{\rho D_o} \hspace{1cm} a^* = \frac{h Y}{\rho D}$$  \hspace{1cm} (7)

Then

$$\frac{A}{A_o} = \frac{D^*}{D^*_o} = \frac{1}{1 - gH/a^*}$$  \hspace{1cm} (8)

By use of equations 1 and 7

$$\frac{a^*_1}{a^*_1 - \frac{h_o D}{h D_o}} = \frac{D^*_1}{D^*_o} = \frac{A}{A_o}$$  \hspace{1cm} (9)

After equating expressions 8 and 9

$$a^* = a^*_1 - gH$$  \hspace{1cm} (10)

"$a^*"$ is the speed of the pressure pulse in the vessel. This statement, that "$a^*"$ decreases with an increase in head, is true if Young's modulus remains constant.\textsuperscript{6}

\textbf{Relation Between Pulse Velocity and Arterial Pressure:} Equation 10, indicating that an increase in arterial pressure causes a decrease in pulse velocity, is at odds with recent reckoning and early experimental evidence.\textsuperscript{11, 18} King\textsuperscript{18} by assigning "elastomeric" properties to the artery wall, developed an equation which indicates that pulse velocity is a positive function of arterial pressure. The fact of the matter is that if the increase in pressure is accompanied by an increase in artery circumference with no change in wall stiffness, then pulse velocity decreases; conversely, if, with the increase in pressure, the wall gets stiffer without an increase in circumference, then pulse velocity increases. It is probable that either or both of these changes occur in the artery, depending on the cause and duration of the increase in arterial pressure. Thus, in the context of our current information, a speculative ramification aimed at exploring the issue of whether pulse velocity increases or decreases with an increase in arterial pressure must consider the specific circumstances under which the pressure increases. The following speculations are illustrative:

1) In an experimental situation in which the active tension of vascular smooth muscle does not change and the intraluminal pressure is increased, it has been well established that the artery wall gets stiffer as it balloons out.\textsuperscript{22} This stiffness would be responsible for an increase in pulse velocity which might overcompensate for the decrease in pulse velocity effected by the increase in circumference. Such a relation between intraluminal pressure and pulse velocity has been observed in isolated vessels\textsuperscript{22} and in man.\textsuperscript{18}

2) In exercise the magnitude and steepness of the pulse pressure increases while the pressure in the arterial system just prior to the systolic upstroke, i.e., diastolic pressure, is not much changed. It is important to note that there is in exercise a several fold increase in the concentration of circulating catecholamines. Since even during rest the blood concentration of these constricators is effective in contracting smooth muscle of the aorta,\textsuperscript{23} increase in catecholamines that occurs with exercise would be expected to cause an increase in active tension in the wall of the aorta. This would make the artery wall stiffer and increase the pulse velocity by a...
PULSATILE PRESSURE AND FLOW

From equation 9

\[ D = \frac{D_{\text{ao}}}{a} \]  

(11)

The speed of the pressure pulse changes both with time \( t \) and with distance \( x \) along the vessel, and whenever "\( a \)" changes reflections are produced.

The partial derivatives of \( A \) with respect to the two independent variables, time \( t \) and distance \( x \) are needed later and result from equations 8 to 11 (a variable subscript \( x \) or \( t \) represents partial differentiation with respect to that variable, i.e., \( A_x = \frac{\partial A}{\partial x} \)).

\[ \frac{A}{A_o} = \frac{a_t}{a^t}, \quad \frac{A_x}{A_o} = -\frac{2a_t}{a^t}a_s, \quad \frac{A_t}{A_o} = -\frac{2a_x}{a^x}a_s \]

mechanism quite unrelated to a change in intraluminal pressure. This stiffness would make the wall less distensible. In this stiffened condition the pressure dependent increase in circumference which is necessary to effect a further increase in stiffness would be small. Therefore, in this artery, stiffened by active contraction of vascular smooth muscle, there should be less of a tendency for Young's modulus to decrease or increase with a given increase in arterial pressure. Pulse velocity would be less influenced by arterial pressure when smooth muscle tone is high than when it is low. In exercise this relationship is complicated by the fact that the vessel is further stiffened by an increased steepness of the pressure pulse. Here the viscous component of wall stiffness is proportionately greater.

3) The increase in arterial pressure associated with chronic hypertension is accompanied by an increase in stiffness of the arterial wall (Bohr, D. F., unpublished observation). This is due primarily to an increase in the non-contractile structural components of the wall. The contribution of active tension from smooth muscle to the total tension in these stiff hypertensive vessels has not been quantified. Again, since these vessels are less distensible the change in circumference for a given change in pressure must be quite small, tending to keep the stress-strain relationship relatively constant over a given pressure range. Changes in circumference and stiffness would be small and therefore the pressure dependent variation in pulse velocity might not be great.

Although it is relatively simple to introduce a pressure dependent Young's modulus into the program currently described, the quantitative nature of this dependence is not known and it is quite likely that the character of the dependence will vary depending on the basis for the pressure change.
which is the continuity equation and must hold throughout the vessel.

4. Momentum Equation

The momentum equation when applied in the x-direction to the fluid in a segmental volume, figure 3, is a statement that the resultant x-component of force on the segment of fluid is just equal to the net efflux of x-momentum plus the time rate of increase of x-momentum within the segment. The resultant force component, figure 3, is

\[ F = PA + (P + P_x \frac{dx}{2}) A_x dx - [PA + (PA_x) dx] - \tau_o \pi D dx \]

The first term is due to pressure within the fluid acting over the cross section at x, the second term is the force component in the x-direction due to the tube wall pushing against the fluid (zero for constant \( A_x \), as \( A_x dx \) is the increase in cross section in the length dx). The term in brackets is the force pushing against the element on the distal side. The action of fluid friction at the tube wall is given by the product of shear stress \( \tau_o \) at the wall and area of wall surface \( \pi D dx \). This force is assumed to act wholly in the x-direction. By expanding the expression for \( F \) it becomes

\[ F = -P_x A dx - \tau_o \pi D dx + P_x A_x (\frac{dx}{2})^2 \]

Since the term with square of dx becomes of a higher order of smallness as dx approaches zero, it may be dropped from the equation.

The momentum influx at x is \( \rho A V^2 \) and the momentum efflux at x + dx is \( \rho A V^2 + (\rho A V^2) dx \), with a net efflux of \( (\rho A V^2) dx \). The time rate of increase of momentum within the segment is given by \( (\rho A V^2) dx \). After combining the force and momentum terms,

\[ -P_x A dx - \tau_o \pi D dx = (\rho A V^2) dx + (\rho A dx V) \]

By expanding the partial derivatives, then dividing through by the mass of the segment \( \rho A dx \), and after replacing \( P_x \) by \( \rho g H_x \),

\[ gH_x + \tau_o \pi D + \frac{A_x}{A} V^2 + 2V V_x + V_x + V \frac{A_t}{A} = 0 \]

(16)

The wall shear stress \( \tau_o \) may be written in the form

\[ \tau_o = k \frac{\rho V^2}{2} \]

(17)

For established, steady laminar flow \( k = 16/R \), with R the Reynolds number \( VD/\mu \), in which \( \mu \) is the fluid viscosity (poise). For turbulent flow \( k = f/4 \), with f the commonly used Darcy-Weisbach friction factor. By inserting equation 17 into equation 16, using f,

\[ gH_x + \frac{fV^2}{2D} + \frac{A_x}{A} V^2 + 2V V_x + V_x + V \frac{A_t}{A} = 0 \]

(18)

This is the momentum equation for flow through a distensible vessel. By multiplying equation 15 by V and subtracting it from...
equation 18, substantial simplification results*

\[ gH_x + VV_x + V_t + \frac{fV_t}{2D} = 0 \]  

(19)

Equations 15 and 19 contain the continuity and momentum principles. After substituting the elastic relationships given by equations 12 into equation 15 it becomes, upon simplification,

\[ VH_x + H_x + \frac{a_t}{g} V_x = 0 \]  

(20)

Equations 19 and 20 are used to develop the final equations.

**5. Development of Characteristic Equations**

By calling equation 19 \( L_1 \) and equation 20 \( L_2 \), they may be combined linearly using an unknown multiplier \( \lambda \), as follows:

\[ L = L_1 + \lambda L_2 = \lambda[H_x(\frac{g}{\lambda} + V) + H_d] \]

If two distinct values of \( \lambda \) are taken, two equations result which contain the momentum and continuity principles. The theory of characteristics determines two special values of \( \lambda \) which result in great simplification of the equations. To review some fundamental relations in calculus, if \( H = H(x, t) \) and \( V = V(x, t) \), then the total derivatives of \( H \) and \( V \) with respect to \( t \) are

\[ \frac{dH}{dt} = H_x \frac{dx}{dt} + H_t, \quad \frac{dV}{dt} = V_x \frac{dx}{dt} + V_t \]

wherein these relationships \( H \) and \( V \) are pressure and velocity of a particle as it moves \( x \) becomes a function of \( t \). By examining equation 21 it is seen that the first bracket contains \( \frac{dH}{dt} \) if

\[ \frac{g}{\lambda} + V = \frac{dx}{dt} \]  

(22)

and the second bracket contains \( \frac{dV}{dt} \) if

\[ V + \frac{\lambda a_t}{g} = \frac{dx}{dt} \]  

(23)

These expressions must be the same. By equating them and solving for \( \lambda \)

\[ \lambda = \frac{g}{a} \]  

(24)

Now, by restricting the applicability of equation 21 to those characteristic lines for which equations 22 and 23 are satisfied, it may be written in the simple form

\[ L = \lambda \frac{dH}{dt} + \frac{dV}{dt} + \frac{fV_t}{2D} = 0 \]  

(25)

By applying \( \lambda = +\frac{g}{a} \) to equations 22 and 25

\[ \frac{dV}{dt} = V + a \]  

(26)

\[ \frac{dH}{dt} = V - a \]  

(27)

Equation 26, a total differential equation, is valid only along a line defined by equation 27, if a plot is made on an \( x - t \) plane, as shown in figure 4. If \( C_+ \) in the figure represents the characteristic line defined by \( dx/dt = V + a \) and passing through a point \( R \) where \( V, H, x \) and \( t \) are known, then equation 26 may be used as one relation between pressure \( H \) and velocity \( V \) along this line. Similarly, equation 28 is valid only along a line defined by equation 29, figure 4. Now with \( V \) and \( H \) known at the known points \( R \) and \( S \), four equations are available at the intersection \( P \) of the two characteristic curves, for computing \( V, H, x \) and \( t \).

**6. Method of Specified Time Intervals**

For computation purposes, equations 26 to 29 are written as finite difference equations.

*Circulation Research, Volume XIII, July 1923*
FIGURE 4
Characteristic lines $C_+$ and $C_-$ drawn through points R and S respectively, where $V$ and $H$ are known. Their intersection $P$ is a point where $x$, $t$, $V'$, and $H$ may be determined from the equations.

For use with a high-speed digital computer, the theory of characteristics method may be extended to a system in which the time and distance intervals are preselected. This is called the method of specified time intervals, and entails interpolation of values of $V$ and $H$ at unknown points R and S, figure 5, such that $P$ occurs at equally spaced distances $Ax$ along the vessel for equal time increments. In figure 5, consider that $V$ and $H$ have been computed for the first two rows at the equally spaced sections. Values of $V$ and $H$ are to be computed next for point $P$ at time $t_P$, which is $t_c + \Delta t$, and values of $V$ and $H$ are known at A, B, and C.

Equations 26 to 29 are written as finite difference equations, for points R and P on $C_+$ and for points S and P on $C_-$, with $\Delta t = t_P - t_R = t_P - t_S$. The mesh ($\Delta x$, $\Delta t$) is assumed to be so fine that the velocity in the friction term may be evaluated at known conditions at C. Also $a_R$ and $a_S$ are replaced by $a_c$, as well as $D_R$ and $D_S$ by $D_c$.

\[ \frac{g}{a_c} (H_R - H_B) + V_F - V_R + \frac{f_c V_c}{2D_c} | V_c | \Delta t = 0 \]  
(30)

\[ V_F - x_R = (V + a)C \Delta t \]  
(31)

\[ - \frac{g}{a_c} (H_F - H_B) + V_F - V_S + \frac{f_c V_c}{2D_c} | V_c | \Delta t = 0 \]  
(32)

\[ V_F - x_B = (V - a)C \Delta t \]  
(33)

The $V^2$ friction term has been replaced by $V/|V|$.

$V_R$, $H_R$, $V_S$, and $H_S$ are now to be computed from the equations, with reference to figure 5, by linear interpolation. The mesh (values of $\Delta x$, $\Delta t$) is assumed to be fine enough so that the slopes of the characteristic lines are given adequately by evaluating $V + a$ and $V - a$ at C. From figure 5

\[ \frac{V_R - V_A}{V_C - V_A} = \frac{x_R - x_A}{x_C - x_A} = \frac{x_R - x_A}{\Delta x} \]  
(34)

By remembering that $x_F = x_C$, from equation 31

\[ x_C - x_R = (V + a)C \Delta t \]  
Then

\[ x_R - x_A = (x_C - x_A) = (V + a)C \Delta t \]  
(35)

For convenience the grid mesh ratio $\Delta t/\Delta x$ is called $\Theta$. Substituting into equation 35, the first of the following four equations is obtained,

\[ V_R = V_A + (V_C - V_A) (1 - \Theta (V + a)C) \]  
(36)

\[ H_R = H_A + (H_C - H_A) (1 - \Theta (V + a)C) \]  
(37)

\[ V_S = V_B + (V_C - V_B) (1 + \Theta (V - a)C) \]  
(38)

\[ H_S = H_B + (H_C - H_B) (1 + \Theta (V - a)C) \]  
(39)

The last three equations are found in a manner similar to that used in obtaining equation 36. In

\[ \Theta = \frac{\Delta t}{\Delta x} \]  
(40)

$\Delta t$ must be selected so that R and S lie within the reach defined by points A and B.

With the interpolated values of $V_R$, $H_R$, $V_S$, $H_S$ known, equations 30 and 32 may now be solved for $H_F$ and $V_F$,

\[ H_F = \frac{H_R + H_S}{2} + a_c \frac{V_R - V_S}{2g} \]  
(41)

\[ V_F = \frac{V_R + V_S}{2} + \frac{g}{2a_c} (H_R - H_S) - \frac{f_c V_c}{2D_c} | V_c | \Delta t \]  
(42)

Equations 36 through 41 permit all interior points of the grid to be computed, i.e., all points where corresponding points, A, B, and C are known. At the end points an additional
condition is needed to solve for \( V_P \) and \( H_P \), since only one of the equations 30 and 32 is available at a given boundary. This additional condition is called the boundary condition.

7. Boundary Conditions

Boundary conditions may take on many forms. Some examples are given here to illustrate procedures in developing them. At the proximal end of the vessel equation 32 applies, figure 6,

\[
V_P = V_S + \frac{g}{a_c} (H_P - H_S) - \frac{f_c V_C | V_C | \Delta t}{2D_c}
\]

in which \( V_P \) and \( H_P \) are the unknowns, as \( V_S \) and \( H_S \) are given by the interpolation equations 38 and 39. If a known pulse inflow \( Q_P \) into the vessel is known at this instant, then a second equation becomes available, as follows:

\[
Q_P = V_P \frac{\pi}{4} D_P \tag{44}
\]

But from equations 10 and 11

\[
D_P = \frac{D_s a_s}{a} = \frac{D_s a_s}{a - gH_P}
\]

By substituting equation 45 into equation 44 to eliminate \( D_P \), and after solving for \( V_P \),

\[
V_P = \frac{4Q_P}{\pi D_s a_s} (a - gH_P) \tag{46}
\]

which is the second relationship required. With the pulse inflow given for each time increment, \( V_P \) and \( H_P \) at the proximal end may be computed progressively as the rest of the solution proceeds. Another example is to have the pressure pulse specified as a function of time at \( x = 0 \). In this case the known \( H_P \) is inserted into equation 43 and \( V_P \) may be computed.

At the distal end, figure 7, the outflow may be expressed in terms of the pressure difference between the vessel \( H_P \) and the terminal bed \( H_B \) as

\[
Q_F = k, (H_P - H_B)^m = V_P \frac{\pi}{4} D_P \tag{47}
\]

equation 30, solved for \( V_P \) is

\[
V_P = V_R - \frac{g}{a_c} (H_P - H_B) - \frac{f_c V_C | V_C | \Delta t}{2D_c} \tag{48}
\]

By use of equations 10 and 11, equation 47 may be solved for \( V_P \) in terms of \( H_B \)

\[
V_P = \frac{4k}{\pi D_s a_s} (H_P - H_B)^m (a - gH_P) \tag{49}
\]

A trial solution may be required, depending upon the value of exponent \( m \) in the terminal bed resistance relationship. Care must be taken in applying a terminal bed \( Q - H \) relation. Within the vessel proximal to the bed, there is no simple \( Q - H \) relation, primarily
due to the complex reflections. If a segment of an arterial system is to be analyzed, the boundary conditions should generally be expressed as \( H \) versus \( t \), \( Q \) versus \( t \), or \( V_P \) versus \( t \).

For known pressure as a function of time, \( V_P \) is found directly from equation 48.

8. Example

To illustrate the application of the characteristics theory to a flow situation, a known pulse flow is injected into a distensible vessel, with a linear terminal bed at the distal end. The computer program, figure 8, is in the MAD (Michigan Algorithmic Decoder) language and an IBM 709 is used in performing the calculations. The pulse flow from the heart of a dog, measured at the ascending aorta by the Square Wave electromagnetic flowmeter,\* has been expressed by a series of empirical formulas. The average flow is 35.65 ml/sec over the 0.4 sec pulse cycle, with a maximum inflow of 160 ml/sec. A friction factor \( f = 0.3 \) was used and a time increment of \( \Delta t = 0.004 \) sec taken with twenty equal reaches of \( \Delta x = 2.5 \) cm. The pressure \( H \), velocity \( V \), diameter \( D \) and flow rate \( Q \) were calculated, and values printed out for every 0.008 sec time interval and at 5 cm distances along the tube. The terminal bed pressure was taken as 100 mm Hg, and the problem was initiated by first setting up a steady flow equal to the average flow (QAVE).

The sequence of main calculations in the program are as follows (fig. 8):

1. Calculation of steady state problem, storing of \( H, V, D, \) and \( Q \) for the 21 sections 2.5 cm apart for time \( t = 0 \).
2. Calculation of interior points, statements 45 through 54, for the next time increment.
3. Calculation of the proximal boundary condition, statements 55 through 67.
4. Calculation of the distal boundary condition equations, statements 68 through 78.
5. Print out of every second set of results, incrementation of \( T \) and \( U \), and check on determination of end of solution.

The program was run through three pulse cycles. The end of the second and third pulses showed rather close agreement, indicating that steady oscillatory flow has almost been established. A gross check on continuity was made as a measure of the accuracy of the finite difference method. The total inflow in the third pulse is \( 35.65 \times 0.4 = 14.26 \) ml. The total outflow is 14.34 ml, and the volume of fluid within the tube has decreased by 0.29 ml. This indicates an error in continuity of about 1.5%, which could be further reduced by taking shorter reaches and smaller time increments, or by going to a method of calculation\* having second order accuracy.

\*Carolina Medical Electronics, Inc., Winston-Salem, North Carolina.
PULSATILE PRESSURE AND FLOW

PULSATILE FLOW THROUGH A DISTENSIBLE TUBE

DIMENSION V(20),H(20),G(20),P(20)

INTEGER I,U,N,P,KK

READ DATA

PRINT FORMAT 2.P.KK

PRELIMINARY CONSIDERATIONS

PULSATILE FLOW

DP=1.E-6

PP=0.2

P=3

QP=4

PRINT P

PRINT FORMAT 2.P.KK

2=P.KK

PRELIMINARY CONSIDERATIONS

PULSATILE FLOW

DP=1.E-6

PP=0.2

P=3

QP=4

PRINT P

PRINT FORMAT 2.P.KK

2=P.KK

PRINT COMMENT

PRELIMINARY CONSIDERATIONS

PULSATILE FLOW

DP=1.E-6

PP=0.2

P=3

QP=4

PRINT P

PRINT FORMAT 2.P.KK

2=P.KK

PRINT COMMENT

PRELIMINARY CONSIDERATIONS

PULSATILE FLOW

DP=1.E-6

PP=0.2

P=3

QP=4

PRINT P

PRINT FORMAT 2.P.KK

2=P.KK

PRINT COMMENT

PRELIMINARY CONSIDERATIONS

PULSATILE FLOW

DP=1.E-6

PP=0.2

P=3

QP=4

PRINT P

PRINT FORMAT 2.P.KK

2=P.KK

PRINT COMMENT

PRELIMINARY CONSIDERATIONS

Figure 8 (part 1) (See legend page 14)

Circulation Research, Volume XIII, July 1963

Downloaded from http://circres.ahajournals.org/ by guest on September 12, 2017
Problem started as a steady state flow with flow of 35.65 ml/sec and head of 14 cm Hg at proximal end. Friction factor $f = 0.3$ and losses varying as square of the velocity.

The calculations are printed out for the third pulse after steady flow, with given for each 0.008 sec time interval (except 0.160 to 0.360 sec).
VESSEL WITH DISTRIBUTED OUTFLOW

ter varying with length along the tube, fig-
the unstressed diameter along its length. The outflow along the
special program has been developed for flow
PULSATILE PRESSURE AND FLOW

Since the vascular system is so complex,
Therefore, for any boundary condition, a

B. EQUATIONS FOR TAPERED DISTENSIBLE VESSEL WITH DISTRIBUTED OUTFLOW

Since the vascular system is so complex, and several computer statements are generally required for any boundary condition, a special program has been developed for flow through a vessel having its unstressed diameter varying with length along the tube, figure 9, and with flow leaving the vessel in a distributed manner. Such a vessel could represent the aorta, with branches of various sizes along its length. The outflow along the vessel is set up as a flow per unit length of vessel, with rate proportional to the head difference inside and outside the tube.

Circulation Research, Volume XIII, July 1963

Figure 8 (part 1)
STEEETER, KEITZER, BOHR

with \( A_x \) a function of \( x \). Then

\[
A_x = A_0 \frac{a_x}{a^*} - 2A_0 \frac{a_x}{a^*} A_x
\]

But

\[
a^* = a_x - gH
\]

and

\[
2a_x a^* = -gH_x
\]

Hence

\[
A_x \frac{A}{A_0} = A_c + gH_x a^* = \frac{\alpha}{A_0} + gH_x a^*
\]

in which \( \alpha \) is the rate of change of unstressed vessel area per unit length. \( A_x / A \) is obtained as before.

Following the previous procedures in developing the finite difference equations, the two controlling equations become (the interpolation equations \( V_R, H_R, V_S, H_S \) are unchanged):

\[
V_P = V_R - \frac{g}{a^* c} (H_R - H_S) - \frac{fV_c}{2D} \frac{|V_c|}{\Delta t} - \frac{a^*}{a^*_c} \frac{V_c}{2D} \frac{\Delta H}{A} - \frac{a^*}{a^*_c} q \frac{\Delta t}{A_{a^*_c}}
\]

\[
V_P = V_S + \frac{g}{a^* c} (H_R - H_S) - \frac{fV_c}{2D} \frac{|V_c|}{\Delta t} + \frac{a^*}{a^*_c} \frac{V_c}{2D} \frac{\Delta H}{A} + \frac{a^*}{a^*_c} q \frac{\Delta t}{A_{a^*_c}}
\]

which simplifies to

\[
V \frac{A_x}{A} + V_x + A_x \frac{A_1}{A} + \frac{q}{A} = 0
\]

By addition, and then by subtraction, the two equations yield values of \( V_P \) and \( H_P \) at interior points:

\[
V_P = \frac{V_R + V_S}{2} + \frac{g}{2a^* c} (H_R - H_S) - \frac{fV_c}{2D} \frac{|V_c|}{\Delta t}
\]

\[
H_P = \frac{H_R + H_S}{2} + \frac{a^*}{2g} \left[ V_R - V_S - \frac{2a^* a^*_c V_c \Delta t}{A_0} - \frac{2a^* q \Delta t}{A_{a^*_c}} \right]
\]

The boundary conditions are handled exactly as before, except that either equation 54 or equation 55 is used, depending upon whether it is a right-end or a left-end bound-

\[
-P_x A_x - r_x \pi D x = (\rho A V^2)_x + (\rho A V d x)_k + q V d x
\]

After reducing this equation in a manner similar to equations 18 and 19, one obtains equation 19 as before.

In order to take the tapering effect into account \( A_x \) is evaluated as follows:

\[
A_x = A_0 \frac{a_x}{a^*} - 2A_0 \frac{a_x}{a^*} A_x
\]
treated as if it were a terminal bed, i.e., a definite relation between pressure $H$ and flow $q$ is established for each branch.

**Comparison of Measured Flow in a Tapering Vessel with Flow Computed from Pressure-Time Data**

To compare flow calculated from experimentally measured pressures with measured flow, an in vivo experiment was performed. Pressure and flow data were obtained from the femoral artery of a dog. Pressure was determined by inserting two short needles in the artery at points 9.4 cm apart. Small branches between were ligated at the vessel wall. Suitable catheters and fittings were connected between the needles and two Sanborn manometers. Flow was measured simultaneously by means of an electromagnetic flow probe placed immediately upstream from the proximal pressure needle. Data were recorded on a four-channel Sanborn (350) system at a chart speed of 100 mm/sec. Figure 10 illustrates the measured proximal and distal pressures obtained for one cycle.

The flow recorded for the same cycle is shown by the solid line in figure 11, and the flow calculated by using the two measured pressures as boundary conditions is shown by the dashed line. Pressure-time values were read off the strip chart for each 0.01 sec. By parabolic interpolation these data were replaced by values for 0.001 sec intervals over the pulse cycle. The 0.001 sec intervals afforded a finer mesh for computation of flow by the method of specified time intervals already described. During the experiment the diameter of the artery and the thickness of the wall were measured with calipers at the two pressure sites. For the computation the diameter of the artery was assumed to decrease linearly along seven equal reaches of the 9.4 cm length. The flow was assumed to be steady initially at about average value, with the head (pressure) constant along the artery. After two cycles have been computed the calculated flow pulse has almost achieved its steady oscillatory character.

The calculated flow does not coincide ex-
Electromagnetic flowmeter data for the upstream section of the femoral artery (fig. 10) and computed flow using the pressure-time data of figure 10 for boundary conditions. In the computer solution the speed of pulse wave at \( H = 11.5 \text{ cm Hg} \) was \( a = 1200 \text{ cm/sec} \). The friction factor was \( f = 0.4 \) and the energy dissipation was taken to vary as the square of the velocity.

Figure 11

Exactly with that measured in vivo. Two groups of factors contribute to this discrepancy: 1) experimental errors, and 2) asymmetry of losses where there is reversal of flow.

Experimental Errors. Pressure measurements are extremely critical. The instantaneous pressure difference between the two ends of this segment of artery ranged between 17.6 mm Hg and -9.2 mm Hg. Errors of 1 mm Hg in locating the zero pressure line can result in a computation that reverses the direction of the average flow. Physical dimensions of the transducer fittings and lines may well introduce their own values and transients that alter the recorded pulse pressures. Amplifier filtering factors in the flowmeter and frequency response of the recording system can modify the contour of the recorded flow. Generally these factors tend to dampen the recorded flow and give rise to symmetrical discrepancies at the crest and nadir of flow pulse. These experimental factors have been studied using programming and the theoretical model. Results of these studies, though not a subject of this paper, have shown that more refined raw data will produce better correspondence between calculated and experimental flow.

Reverse Flow Friction Factor. It is evident in figure 11 that there is also an asymmetrical discrepancy between the calculated and recorded flow pulses. Calculated backflow is much greater than the recorded backflow, whereas positive flow pulses correspond quite well. The following hypothesis is proposed to account for this particular asymmetry which is a consistent finding.

The friction term from equation 18 \( \frac{fV^2}{2D} \) indicates that the energy losses due to shearing forces at the vessel wall are a function of \( f \) and velocity squared. As noted after equation 17, the resistance constant for steady laminar flow is \( k = 16/R \), and for turbulent flow \( k = f/4 \). Substituting the laminar statement results in calculated flows that are much greater than our measured flows, demonstrating that the resistance of laminar flow (as expressed by Poiseuille's equation) is not large enough where actual proximal and distal pressure-time data are used as boundary conditions in the femoral artery. Although it is generally held that arterial flow tends to be laminar, the energy dissipation due to friction is not adequately expressed by steady flow friction equations. The energy losses for unsteady flow seen in the larger arteries can be evaluated better by turbulent friction relationships.

The nature of pulsatile flow in tapered vessels is such that a steady state laminar flow is never attained. For the example illustrated here the pressure gradient reverses during the cycle and the reverse pressure gradient will effect a separation of the boundary layer. This separation causes vortices at the wall which spread through the tube giving rise to added energy losses. Flow in a converging tube tends to produce a thin boundary layer and the resulting losses are less than where the flow is in a diverging tube (flow backward in a tapering vessel). Thus the losses occur-

Circulation Research, Volume XIII, July 1963
ring in pulsatile flow will be asymmetrical; that is, the losses where the flow is in a backward direction will be greater than those where the flow is forward.

In making the calculations for the illustration here the friction term was expressed as \( f \frac{V}{2D} \frac{V}{V} \) \( ^{-1} \). By a gradient method, values for \( f \) and \( n \) were obtained that gave the best fit (least square method) with the flowmeter data. These results yielded values for \( f \) of about 0.4 and for \( n \) about 2.0. These values for \( f \) and \( n \) are not great enough to express the increased losses when the flow reverses. If this hypothesis is correct, this asymmetrical discrepancy between the calculated and recorded backward flows would be correctable by finding appropriately larger values for \( f \) and \( n \) when the sign of the friction changes with reversal of the flow.

**Summary**

The basic differential equations for elastic wall material and for continuity and momentum are derived, including fluid frictional resistance of the wall of the tubes, based on one-dimensional flow. These partial differential equations are transformed into four ordinary differential equations using the theory of characteristics. Then difference equations are developed and by an interpolation method (method of specified time intervals) equations are obtained for computation of velocity and pressure at equally-spaced sections along the vessel at specified equal time intervals. The equations are first applied to a flexible tube of initial constant diameter, with a pulse flow taken from in vivo experiments.

Equations are then developed for tapering tubes with distributed outflow along their lengths (to simulate branches). Pressure-time data from femoral artery measurements are then used to compute flow through the artery, and the results are compared with electromagnetic flowmeter data. Computed flow is greater than measured flow if a friction factor for laminar flow is used. Frictional losses of energy in the normal pulsatile flow in the femoral artery are similar to those of turbulent flow.

**Acknowledgment**

The authors are indebted to the University of Michigan Computing Center for use of its digital computer.

**References**


### Appendix

**Symbols**

\[ x = \text{distance along vessel} \]
\[ Y = \text{elastic modulus of vessel wall} \]
\[ a = \text{rate of change of unstressed vessel area per unit length} \]
\[ \Delta t = \text{ratio of change of unstressed vessel area} \]
\[ \lambda = \text{undetermined multiplier} \]
\[ \mu = \text{dynamic viscosity} \]
\[ \rho = \text{density of fluid} \]
\[ \tau_s = \text{frictional wall shear stress} \]

**Notation for Michigan Algorithmic Decoder (MAD) Program**

\[ A = \text{speed of pressure pulse at pressure } H_{0} \]
\[ AC = \text{speed of pressure pulse at pressure } H \]
\[ AO = \text{speed of pressure pulse at zero pressure} \]
\[ ACM = \text{speed of pressure pulse at pressure } H(N) \]
\[ C1, C2, C3, C4, C5, C6, C7 = \text{constants} \]
\[ D = \text{diameter} \]
\[ DELT = \text{time increment} \]
\[ DF = \text{steady state pressure drop in reach } \]
\[ F = \text{friction factor} \]
\[ G = \text{gravity} \]
\[ H = \text{pressure, from previous calculation} \]
\[ HO = \text{initial pressure} \]
\[ HB = \text{bed pressure} \]
\[ HP = \text{pressure to be calculated} \]
\[ HR = \text{interpolated head} \]
\[ HS = \text{interpolated head} \]
\[ I = \text{integer} \]
\[ K = \text{constant in terminal bed relation} \]
\[ KK = \text{constant} \]
\[ N = \text{number of reaches} \]
\[ P = \text{constant} \]
\[ PMT, PPT, PTT = \text{constants in computing pulse} \]
\[ PTIME = \text{pulse period} \]
\[ Q = \text{flow} \]
\[ QAVE = \text{steady-state flow} \]
\[ QM = \text{maximum pulse flow} \]
\[ RHO = \text{density} \]
\[ SG = \text{specific gravity of mercury in terms of fluid flowing} \]
\[ T = \text{time} \]
\[ TM = \text{constant in computing pulse} \]
\[ U = \text{integer} \]
\[ V = \text{velocity, from previous calculation} \]
\[ VP = \text{velocity to be calculated} \]
\[ VR = \text{interpolated velocity} \]
\[ VS = \text{interpolated velocity} \]